HANCOCK'S APPLIED MECHANICS FOR ENGINEERS

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HANCOCK'S APPLIED MECHANICS FOR ENGINEERS

REVISED AND REWRITTEN

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PREFACE

In the preparation of this book the author has had in mind the fact that the student finds much difficulty in seeing the applications of theory to practical problems. For this reason each new principle developed is followed by a number of applications. In many cases these are illustrated, and they all deal with matters that directly concern the engineer. It is believed that problems in mechanics should be practical engineering work. The author has endeavored to follow out this idea in writing the present volume. Accordingly, the title "Applied Mechanics for Engineers" has been given to the book.

The book is intended as a text-book for engineering students of the Junior year. The subject-matter is such as is usually covered by the work of one semester. In some chapters more material is presented than can be used in this time. With this idea in mind, the articles in these chapters have been arranged so that those coming last may be omitted without affecting the continuity of the work. The book contains more problems than can usually be given in any one semester.

While it is difficult to present new material in the matter of principles, much that is new has been introduced in the applications of these principles. The subject of Couples is treated by representing the couples by means of vectors. The author claims that the chapters on Moment of Inertia, Center of Gravity, Work and Energy, Friction and Impact are more complete in theory and applications than those of any other

American text-book on the same subject. These are matters upon which the engineer frequently needs information; frequent reference is, therefore, given to original sources of information. It is hoped that these chapters will be especially helpful to engineers as well as to students in college, and that they will receive much benefit as a result of looking up the references cited. In general, the answers to the problems have been omitted for the reason that students who are prepared to use this book should be taught to check their results and work independently of any printed answer.

The author wishes to acknowledge the helpful suggestions obtained from the many standard works on mechanics. An attempt has been made to give the specific reference to the original for material taken from engineering works or periodical literature. He wishes, moreover, to express his thanks to Dean C. H. Benjamin and Professor L. V. Ludy for their careful reading of the manuscript, to Professor W. K. Hatt for many of the problems used, and to Dean W. F. M. Goss, whose continued interest and advice have been a constant source of inspiration. It is hoped that the work may be an inspiration to students of engineering.

E. L. HANCOCK.

PURDUE UNIVERSITY,
November, 1908.

PREFACE TO THE REVISED EDITION

IN the revision of this text, although rather extensive changes in method of treatment have been made in certain parts, the general subject matter and order of arrangement have been, in the main, retained. The chapter on Dynamics of Machinery, now called Dynamics of a Rigid Body, has been transferred to a later position in the book on account of its relatively greater difficulty.

An important change is to be found in the much larger use of graphical methods. The graphical and analytical methods have been developed simultaneously and many problems are given to be solved by both methods. The use of graphical methods has brought about the introduction of considerable new material, particularly in the construction of stress diagrams for trusses, and in the application of the equilibrium polygon to centers of gravity of plane areas, to weighted strings and linkages, and to bending moment diagrams.

The problems illustrating the principles follow immediately after the development of the principles but are of such a nature as to require original thinking. The solution does not consist in finding and substituting in the proper formula. About two hundred new problems have been added throughout the book.

I wish to express my thanks to Mr. E. G. Frazer for a careful reading of the proof and for helpful suggestions.

N. C. RIGGS.

SCHOOL OF APPLIED SCIENCE,
CARNEGIE INSTITUTE OF TECHNOLOGY.
January, 1915.

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APPLIED MECHANICS FOR ENGINEERS

CHAPTER I

DEFINITIONS

1. Introduction. — The study of the subject of mechanics involves a study of *matter*, *space*, and *time*, and of the behavior of bodies under the action of forces. The subject as presented in this book consists of two parts; viz. *statics*, including the study of bodies under the action of systems of forces that are in equilibrium (balanced), and *dynamics*, including a study of the motion of bodies.

2. Force. — A body acted upon by the attraction or repulsion of another body is said to be subjected to an attractive or repulsive *force*, as the case may be. In simple terms a force is a *push* or a *pull*. Forces are usually defined by the effects produced by them, as, for example, we say, a force is something that produces motion or tends to produce motion, or changes or tends to change motion, or that changes the size or shape of a body. Forces always occur in pairs; for example, a book held in the outstretched hand exerts a downward pressure on the hand, and the hand exerts an equal upward pressure on the book.

3. Inertia. Mass. — It is a matter of common experience that bodies vary in the amount of resistance that they offer to a change of their state of rest or motion. Thus

it is more difficult to stop a swiftly moving ball of iron than one of wood having the same dimensions and speed. It is easier to set in motion the wooden ball than to give the same speed to the iron ball of the same size.

The property of resistance to change of its state of rest or motion that every body has is called its *inertia.*

The word *mass* is used in the sense of a measure of inertia.

4. The Unit of Mass. — A certain arbitrary piece of platinum carefully preserved by the British government is known as the standard mass of one *pound avoirdupois.* Any other piece of matter which under the action of a certain force has its motion changed in the same way as the standard mass of one pound under the same conditions is also called a mass of one pound.

The corresponding unit in the French system is the *gram.*

5. The Unit of Force. — The pull, or attraction, that the earth exerts upon a mass of one pound at sea level and latitude 45° is called the force of one *pound.* Similarly, the force of one *gram* is the force that the earth exerts upon a mass of one gram at sea level and latitude 45°. These units of force are called the *gravitational units of force.*

The *poundal* is defined as that force which acting alone on a mass of one pound for one second produces in that mass a change of velocity of one foot per second.

The *dyne* is defined as that force which acting alone on a mass of one gram for one second produces in that mass a change of velocity of one centimeter per second.

The poundal and dyne are called *absolute units of force.*

The force with which the earth attracts a body is called

the *weight* of the body. Mass differs from weight, in that the weight varies with the position on the surface of the earth and with the height above the surface, while the mass remains the same. The weight of a body may be determined by means of the spring balance. The equal-armed balance gives the same weight regardless of distance from the center of the earth. The equal-armed balance really measures mass.

6. Unit Weight. — The weight of a cubic foot of a substance will be called the *unit weight* of the substance and will be represented by γ. Below is given a table of such

TABLE I

UNIT WEIGHTS AND SPECIFIC GRAVITY OF SOME MATERIALS

(Kent's "Engineer's Pocket Book")

MATERIAL	SPECIFIC GRAVITY	UNIT WEIGHT
Brick		
Soft	1.6	100
Hard	2.0	125
Fire	2.24–2.4	140–150
Brickwork — mortar	1.6	100
Brickwork — cement	1.79	112
Concrete	1.92–2.24	120–140
Copper	8.85	552
Earth — loose	1.15–1.28	72–80
Earth — rammed	1.44–1.76	90–110
Iron — cast	7.21	450
Iron — wrought	7.7	480
Masonry — dressed	2.24–2.88	140–180
Pine — white	.45	28
Pine — yellow	.61	38
Steel	7.85	490
White Oak	.77	48

weights at sea level. The unit weight of a substance divided by the unit weight of pure water gives its *specific gravity*.

7. Rigid Body. — In studying the state of motion or rest of a body due to the application of forces acting upon it, the deformation of the body itself, due to the forces, may be disregarded. When so considered, it is customary to say that the body is a *rigid body*. Unless otherwise stated bodies will be considered as rigid bodies in this book.

8. Vectors. — Any quantity that has magnitude and direction may be represented graphically by a directed segment of a straight line. The length of the segment is taken to measure the magnitude of the quantity, and the direction of the segment to indicate the direction of the quantity.

The directed segment of the line is called a *vector*, and the quantity it represents, a vector quantity.

9. Displacement. — By the displacement of a body is meant its change from one position to another.

Since a change of position involves magnitude and direction, a displacement may be represented by a vector, the length of the vector representing the distance from the first position to the second and the direction of the vector representing the direction of the second position from the first.

From the definition of a displacement it follows that two successive displacements are equivalent to a single displacement. Thus, if a man walks due east one mile and then due north one mile, we might represent his dis-

placement from the original position by a vector drawn northeast to represent a length equal to $\sqrt{2}$ miles. Or, in Fig. 1, if P_1 represents a displacement of a body in the direction indicated and P_2 a subsequent displacement in the direction of P_2, then R represents a displacement equivalent to P_1 and P_2. It is seen that R may be determined by constructing a parallelogram on P_1 and P_2 as sides and drawing the diagonal.

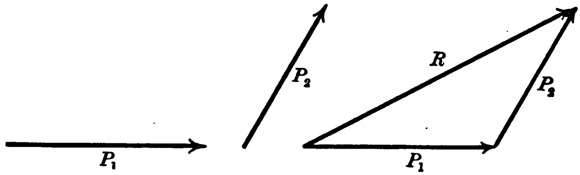


FIG. 1

10. Representation of Forces. — A force has a certain magnitude, acts in a certain direction, and has a definite point of application. If a man, for example, attaches a rope to a log and pulls on the rope, his pull may be measured in pounds, it acts along the rope, and its point of application is the point of attachment of the rope to the log. It is convenient, for the purpose of analysis, to represent forces by vectors, the length of the vector representing the magnitude of the force and its direction giving the direction in which the force acts. Thus, a 10-lb. force, acting in a direction 30° with the horizontal, is represented by a vector drawn in the same direction and having its point of application in the body, and having a length representing 10 lb. (In this case, if 1 in. represents 2 lb., the length of the vector is 5 in.) The line along which a force acts will be referred to as its *line of action.*

11. Transmissibility of Forces. — It is a matter of experience that the point of application of a force may be changed to any point along its line of action without changing the effect of the force upon the rigid body. This, of course, is on the assumption that all the force is transmitted to the body. The law may be stated as follows: *The point of application of a force may be transferred anywhere along its line of action without changing its effect upon the body upon which it acts.*

12. Concurrent Forces. — When the lines of action of two or more forces pass through the same point, the forces are known as *concurrent forces.*

13. Resultant of Two Concurrent Forces. — If two concurrent forces act on a body, there is some single force that might be applied at the point of intersection of the forces to produce the same effect. This single force is called the *resultant* of the two forces. Experiment shows that it may be found as follows: construct upon the vectors representing the forces, laid off from the intersection of their lines of action, a parallelogram and draw the diagonal from the point of application. This diagonal represents the resultant of the two forces in magnitude and direction and it is the line of action of the resultant.

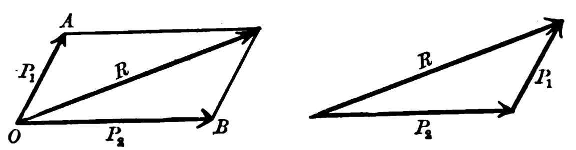


FIG. 2

Thus, if P_1 and P_2 (Fig. 2) are the forces, then R is the resultant.

Algebraically $R = \sqrt{P_1^2 + P_2^2 + 2\,P_1P_2 \cos AOB}$.

Instead of speaking of the vector which represents a force, we shall for the sake of brevity speak of the vector as the force.

14. Resolution of Force. — We have just seen how two concurrent forces may be replaced by a single force called their resultant. In a similar way a single force may be resolved into two forces. These forces are the sides of a parallelogram of which the single force is a diagonal. It is clear, then, that there are an infinite number of components into which a single resultant may be resolved. It is necessary, therefore, in speaking of the components of a force, to state specifically which are intended. It will be seen in problems that follow that the components most often used are at right angles to each other, and are usually the *vertical* and *horizontal* components. In such a case the components are the *projections of the force on the vertical and horizontal lines.*

15. Force Triangle. — It follows directly from the parallelogram law of forces that if we draw from any point a line parallel and equal to one of two concurrent forces, P_2 say, and from the extremity of this line another line parallel and equal to P_1, then the remaining side of the triangle will represent the resultant R. This triangle is called the *force triangle.* In general, the resultant of two concurrent forces may be found by drawing lines parallel to the forces as above. The line necessary to complete the triangle is the resultant, and its direction is always away from the point of application. The equal and opposite

of this resultant would be a single force that would hold the two concurrent forces in equilibrium.

The single force which will balance a given set of forces is called their *equilibrant*.

16. Force Polygon. — If more than two forces are concurrent, we may find their resultant by proceeding in a way similar to that outlined above. Thus, let the forces be P_1, P_2, P_3, P_4, etc. (Fig. 3), all passing through a

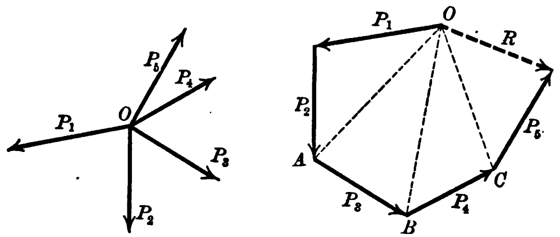


FIG. 3

point; from any point draw a line equal and parallel to P_1, from the extremity of the line draw another equal and parallel to P_2, from the extremity of this last line draw another equal and parallel to P_3, and proceed in the same way for the other forces. The figure produced will be a polygon whose sides are equal and parallel to the forces. The resultant of the given forces is then represented in magnitude and direction by the line necessary to close the polygon, and its line of action passes through the intersection of the given concurrent forces.

The arrow, representing the direction of the resultant, will always be away from the point of application. (See Fig. 3.)

By drawing the lines OA, OB, OC, etc., it is easy to

see that OA represents the resultant of P_1 and P_2, that OB represents the resultant of OA and P_3, and so of P_1, P_2, and P_3, etc. That is, the force polygon follows directly from the force triangle. If the polygon be closed, the system of forces will be in equilibrium, and conversely. The single force necessary to produce equilibrium will, in any case, be equal and opposite to the resultant R. The student should construct force polygons by taking the forces in different orders and checking the resultant in each case.

By means of the force polygon it is easy to find graphically the resultant of any number of concurrent forces in a plane. The work, however, must be done accurately.

The student should show that the force polygon may be used for finding the resultant of concurring forces in space, by considering two forces at a time. The force polygon in this case is called a *twisted polygon.*

Problem 1. Find graphically the resultant in magnitude, direction, and point of application of the following four concurrent forces in one plane: 80 lb. in direction E., 60 lb. E. 50° N., 100 lb. W. 40° N., and 120 lb. E. 30° S. Check by taking the forces in different orders.

Problem 2. Find graphically the resultant of the following four concurrent forces: (*a*) 50 lb. directed E. in a horizontal plane, (*b*) 80 lb. directed N. 40° W. in the horizontal plane, (*c*) 100 lb. directed 30° above the horizontal, the projection of the force on the horizontal plane being directed S. 20° E., and (*d*) 70 lb. directed 60° above the horizontal, the projection of the force on the horizontal plane being directed W. 40° S.

Suggestion. Resolve the forces (*c*) and (*d*) into their horizontal and vertical components before combining with the other forces.

CHAPTER II

CONCURRENT FORCES

17. Concurrent Forces in a Plane. — It will often be convenient to consider forces as acting on a material point; this is equivalent to considering the mass of the body *as concentrated at* a point. Given a material point O (Fig. 4) acted upon by a number of forces in a plane, P_1, P_2, P_3, P_4, etc., making angles α_1, α_2, α_3, α_4, etc., respectively, with the positive x-axis, it is desired to find the resultant of all of them in magnitude and direction; that is, the single ideal force that would produce the same effect as the system of forces.

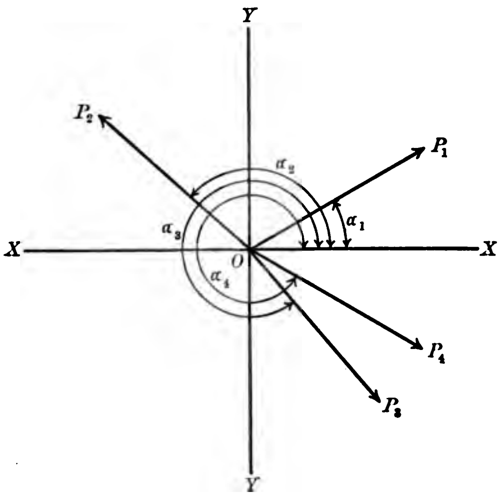


FIG. 4

Each force P may be resolved into components along the x- and y-axes, giving $P \cos \alpha$ along the x-axis, and

$P \sin \alpha$ along the y-axis. The sum of these components along the x-axis may be expressed,

$$\Sigma X = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \text{etc.},$$

the proper algebraic sign being given $\cos \alpha$ in each case. In a similar way the sum of the components along the y-axis may be written,

$$\Sigma Y = P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 + \text{etc.}$$

These forces, ΣX and ΣY, may now replace the original system as shown in Fig. 5; and these may be combined

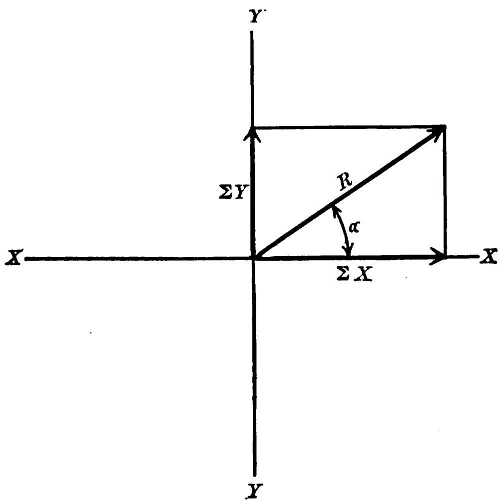


FIG. 5

into a single resultant which is the diagonal of the rectangle of which the two forces are sides (Art. 15). This gives the resultant in magnitude and direction, and this resultant force is the single force which, if allowed to act upon the material point, would produce the same effect as

the system of forces. Analytically the resultant may be expressed,

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2},$$

and its direction makes an angle α with the x-axis such that $\tan \alpha = \frac{\Sigma Y}{\Sigma X}$. (See Fig. 5.) If the forces are in equilibrium, this resultant force must be equal to zero; that is,

$$\sqrt{(\Sigma X)^2 + (\Sigma Y)^2} = 0.$$

This means that $(\Sigma X)^2 + (\Sigma Y)^2 = 0$, that is, that the sum of two squares must be zero; but this can happen only when each one, separately, is zero (since the square of a real quantity cannot be negative). We therefore have as the necessary and sufficient conditions for the equilibrium of a material point, acted upon by a system of concurring forces in a plane,

$$\Sigma X = 0, \text{ and } \Sigma Y = 0.$$

When R is not zero, the system of forces causes accelerated motion in the direction of R; when $R = 0$, the material point remains at rest, if at rest, or continues in motion with uniform velocity, if in motion. In this case the system of forces is said to be *balanced*.

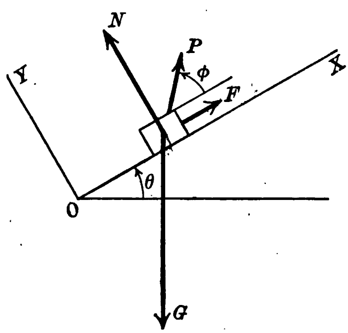


FIG. 6

As an illustration of the foregoing, consider the case of a body of weight G situated on an inclined plane, making an angle θ with the horizontal. (See Fig. 6.) There is a certain force P, making an angle ϕ with the plane, whose component along

the plane acts upwards, and also a force of friction F upwards. The other forces acting on the body are G, the force of gravity acting vertically, and N, the normal pressure of the plane. Taking the x-axis along the plane positive upward and the y-axis perpendicular to it positive upward, we get,

$$\Sigma X = P\cos\phi + F - G\sin\theta,$$

and

$$\Sigma Y = N + P\sin\phi - G\cos\theta.$$

For equilibrium

$$P\cos\phi + F - G\sin\theta = 0,$$
$$N + P\sin\phi - G\cos\theta = 0.$$

Therefore, $N = G\cos\theta - P\sin\phi,$

$$P = \frac{G\sin\theta - F}{\cos\phi}.$$

This last equation gives the magnitude of P required to preserve equilibrium, supposing that the force of friction, θ, and ϕ are known.

Problem 3. Given three concurring forces, 100 lb., 50 lb., and 200 lb., whose directions referred to the x-axis are 0°, 60°, 180°, respectively; find the resultant in magnitude and direction.

Problem 4. A body (Fig. 7) whose weight is G is drawn up the inclined plane with uniform velocity due to the action of the forces P and P'. Find the force of friction F, and the normal pressure N, if $P = 100$ lb., $P' = 100$ lb., $G = 160$ lb. P acts parallel to the plane and P' acts horizontally.

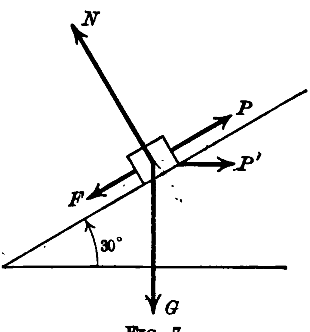


FIG. 7

18. A Body in Equilibrium under the Action of Two Forces. Tension, Compression. — Two forces are in equilibrium when and only when their lines of action coincide and one of the forces is equal in value but opposite in direction to the other. In roof and bridge trusses, in cranes, and in other jointed structures all forces acting on each member are often considered as applied at two points near the ends of that member, and the member may thus be regarded as acted upon by only two forces. Where such is the case the two forces acting upon the member must be equal and opposite and their lines of action must coincide with the line joining the points of application of the forces.

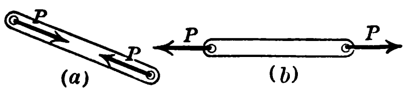


FIG. 8

If the two forces act towards each other, the member is said to be in *compression*. If the forces act away from each other, the member is said to be in *tension*. Thus in Fig. 8 the member is in compression in (*a*) and in tension in (*b*), the amount of compression or tension being P.

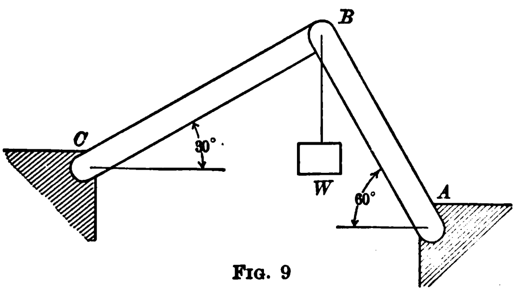


FIG. 9

Illustration. In Fig. 9 a horizontal pin carrying the weight W passes through the members AB and BC which

are inclined respectively at angles 60° and 30° to the horizontal and held at *A* and *C* by horizontal pins perpendicular to the plane *ABC*. If the weights of *AB* and *BC* are small compared to *W*, *AB* and *BC* may each be regarded as acted upon by only two forces. The values of these forces may be found as follows: Since *AB* and *BC* are in equilibrium under the action of only two forces, those acting at the pins, the forces acting on each member

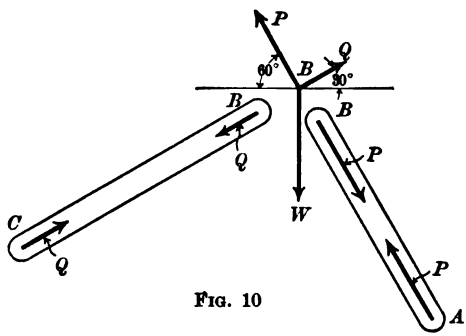


FIG. 10

must be along the lines joining the pins. The members themselves must then exert on the pins forces equal and opposite to those that the pins exert on the members. Hence acting on the pin at *B*, holding it in equilibrium, are the forces *W*, *P*, and *Q*, acting respectively downward, along *AB*, and along *CB* (Fig. 10). Applying the conditions for equilibrium, $\Sigma X = 0$, $\Sigma Y = 0$, there results

$$P \cos 60° - Q \cos 30° = 0,$$

$$P \sin 60° + Q \sin 30° - W = 0.$$

Solving these equations, we find

$$P = .866\ W,$$

$$Q = .5\ W.$$

A graphical solution is obtained by laying out the force triangle for the forces in equilibrium at the point B and measuring the sides representing P and Q (Fig. 11).

Problem 5. A weight of 10 tons is supported as shown in Fig. 12. Find the force acting in the tie A and the member B.

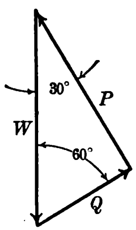


FIG. 11

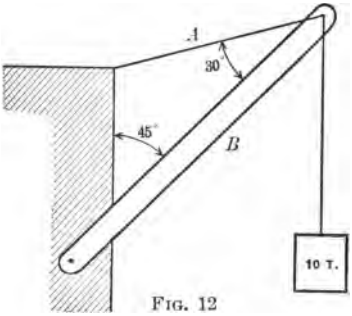


FIG. 12

Problem 6. Two ropes of length 5 ft. and 8 ft. are attached at one end to a weight of 500 lb. The other ends of the rope are attached to two points 9 ft. apart on a horizontal line. When the weight is suspended by the ropes, find the tension in each rope.

19. Body in Equilibrium under the Action of Three Forces. — *If three forces are in equilibrium, any one of them must be the equilibrant of the other two. Its line of action must therefore pass through the intersection of the lines of action of the other two.* Also the vector of the third force must be equal and opposite to the diagonal of the parallelogram formed on the vectors of the other two forces as sides. Imposing these conditions on any body in equilibrium under the action of three forces will usually suggest an easy method of finding two of the forces acting on the body when the third one is known.

Illustration. A gate 9 ft. wide by 5 ft. high, weighing 80 lb., is hung by two hinges distant respectively 6 in. from the top and the bottom of the gate. If the lower hinge carries all of the weight (*i.e.* the upper hinge exerts no upward force on the gate), find the reactions of the hinges. The weight of the gate may be assumed to act at the center.

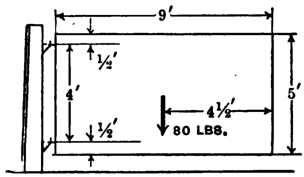


FIG. 13

SOLUTION. The gate is acted upon by three forces: the weight, 80 lb., acting downward at the center, a horizontal force exerted by the upper hinge, and a force unknown in magnitude and direction exerted by the lower hinge. These three forces must pass through a common point, the point A, Fig. 14, and form a closed triangle of forces. Hence, if $AB = 80$ lb., the forces P and Q in Fig. 14 represent on the same scale the reactions of the lower and upper hinges respectively. The values may be found by measurement, or by solving the triangle ABC. Comparing the similar triangles ABC and AED, one a triangle of forces, the other a triangle of distances, we have

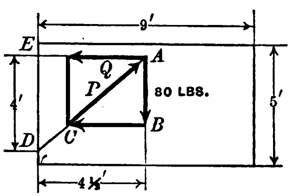


FIG. 14

$$\frac{Q}{80} = \frac{4.5}{4}. \qquad \therefore\ Q = 90 \text{ lb.}$$

$$\frac{P}{80} = \frac{\sqrt{4^2 + (4.5)^2}}{4}. \qquad \therefore\ P = 120.4 \text{ lb.}$$

Problems of this nature are more easily solved by use of the theory of moments developed later.

Problem 7. Neglecting the weight of the members *AD* and *BC* in Fig. 15, find analytically and graphically the stress in *BC* and the reaction at *A*.

Suggestion. *BC* is acted upon by two forces, one at *B* and one at *C*. *AD* is acted upon by three forces. For the analytical solution first find the values of *AO* and *BO* and compare the force triangle with the triangle *AOB*.

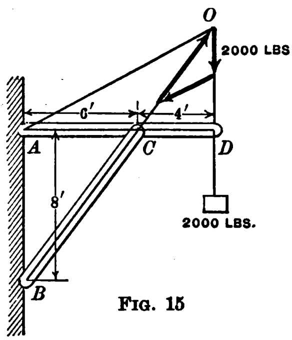


Fig. 15

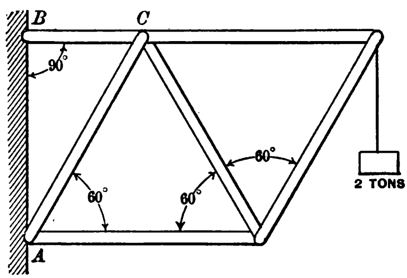


Fig. 16

Problem 8. A post 20 ft. high is hinged at the foot and stands vertically. Two ropes *BC* and *ED* on opposite sides of the post and in a plane perpendicular to the axis of the hinge make angles of 50° and 40° respectively with the horizontal and are attached to the post at distances of 10 ft. and 20 ft. respectively from the bottom. What tension in *ED* will cause a tension of 500 lb. in *BC*, and what will then be the reaction at the foot of the post? Solve graphically.

Problem 9. Neglecting the weight of the truss in Fig. 16, find analytically and graphically the reaction at *A* and the tension in *BC*.

Problem 10. Find analytically and graphically the value of the horizontal force *P* that will just raise the corner *A* of the block in Fig. 17 from the floor. Find also, graphically, the value of *P* when *A* is raised 6 in. from the floor. When *A* is raised 1 ft.

Problem 11. The column AB, Fig. 18, is one foot square, 15 ft. long, and weighs 1200 lb. Find graphically the tension in the rope and the reaction at A when the angle which the column makes with

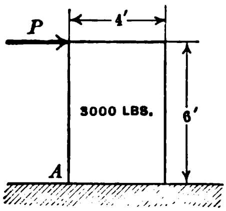


Fig. 17

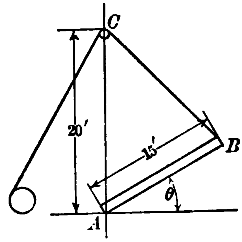


Fig. 18

the horizontal is 0°, 30°, 60°. The pulley at C is small. Consider the weight of the column as acting at its center.

Problem 12. A wheel is about to roll over an obstruction. The diameter of the wheel (Fig. 19) is 3′ and its weight 800 lb. Find the horizontal force P through the center necessary to start the wheel over the obstruction.

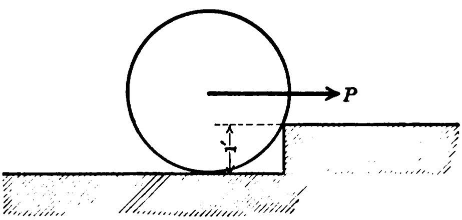


Fig. 19

Problem 13. An angle iron, whose weight is 20 lb. and angle a right angle, rests upon a circular shaft, radius 2 in. Find the normal pressure at A and B (Fig. 20).

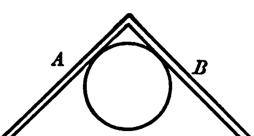


Fig. 20

20. Concurrent Forces in Space. — Let P_1, P_2, P_3, etc. be a set of concurrent forces not in one plane, and let their

direction angles be respectively α_1, β_1, γ_1; α_2, β_2, γ_2; α_3, β_3, γ_3; etc. (The direction angles of a force are the

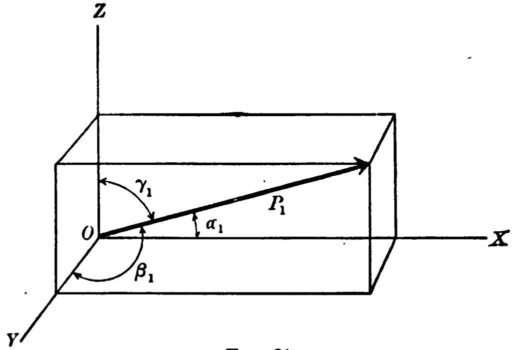


FIG. 21

angles at the point of application of the force measured from the positive direction of the coördinate axes to the vector representing the force.) The force P_1 may be resolved into components parallel to the coördinate axes,

$$X_1 = P_1 \cos \alpha_1,\ Y_1 = P_1 \cos \beta_1,\ Z_1 = P_1 \cos \gamma_1.$$

The resultant force may be found in magnitude and direction by an analysis similar to that used in Art. 17. The sum of the components of all the forces parallel to the x-axis is

$$\Sigma X = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \text{etc.},$$

the sum of the components parallel to the y-axis,

$$\Sigma Y = P_1 \cos \beta_1 + P_2 \cos \beta_2 + P_3 \cos \beta_3 + \text{etc.},$$

and the sum of the components parallel to the z-axis,

$$\Sigma Z = P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + P_3 \cos \gamma_3 + \text{etc.}$$

The original system of forces may now be replaced by a system of three rectangular forces ΣX, ΣY, and ΣZ

(Fig. 22). Finally, this system may be replaced by a resultant which is the diagonal of a parallelopiped constructed with ΣX, ΣY, ΣZ as edges. In magnitude this resultant may be expressed

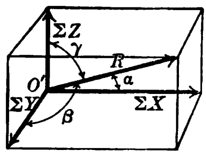


FIG. 22

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}$$

(see Fig. 22) and its direction given by the angles α, β, and γ. These angles are given by the equations

$$\cos\alpha = \frac{\Sigma X}{R},\ \cos\beta = \frac{\Sigma Y}{R},\ \cos\gamma = \frac{\Sigma Z}{R}.$$

For equilibrium R must be 0; that is,

$$(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2 = 0,$$

and therefore,

$$\Sigma X = 0,\ \Sigma Y = 0,\ \Sigma Z = 0.$$

This gives three equations of condition from which three unknown quantities may be determined. In the preceding case of Art. 17 there were only two equations of condition $\Sigma X = 0$ and $\Sigma Y = 0$; consequently, only two unknown quantities could be determined.

Problem 14. Prove that if α, β, γ are the direction angles of any straight line, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Problem 15. Three men (Fig. 23) are each pulling with a force P at the points a, b, and c, respectively. What weight Q can they raise with uniform motion if each man pulls 100 lb.? Each force makes an angle of 60° with the horizontal and the projections of the forces on a horizontal plane make angles of 120° with each other. Solve analytically and also graphically by projecting each force on the vertical and horizontal planes.

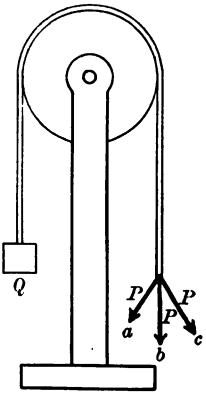


FIG. 23

Problem 16. Three concurring forces act upon a rigid body. Find the resultant in magnitude and direction. The forces are defined as follows:

$P_1 = 75$ lb.; $\alpha_1 = 63° 27'$; $\beta_1 = 48° 36'$; $\gamma_1 = ?$
$P_2 = 80$ lb.; $\alpha_2 = 153° 44'$; $\beta_2 = 67° 13'$; $\gamma_2 = ?$
$P_3 = 95$ lb.; $\alpha_3 = 76° 14'$; $\beta_3 = 147° 2'$; $\gamma_3 = ?$

HINT. γ_1, γ_2, and γ_3 may be found from the relation:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Problem 17. Each leg of a pair of shears (Fig. 24) is 50 ft. long. They are spread 20 ft. at the foot. The back stay is 75 ft. long. Find the forces acting on each member when lifting a load of 20 tons at a distance of 20 ft. from the foot of the shear legs, neglecting the weight of the structure.

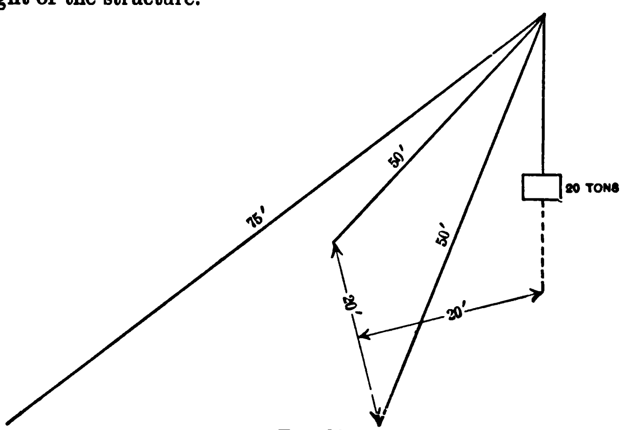


FIG. 24

21. Moment of a Force. — *The moment of a force with respect to any point is defined as the product of the force and a perpendicular from the point to the line of action*

of the force. Let P (Fig. 25) be the force and O the point and a the perpendicular distance of the force from the point; then Pa is the moment of the force with respect to the point O. This moment is measured in terms of the units of both *force* and *length*, viz. pound-feet or pound-inches, and is read pound-feet or pound-inches to distinguish it from foot-pounds of work or inch-pounds of work.

For convenience the algebraic sign of the moment is said to be *positive* when the moment tends to turn the body in a direction *counter-clockwise*, and *negative* when it tends to turn the body in the *clockwise* direction.

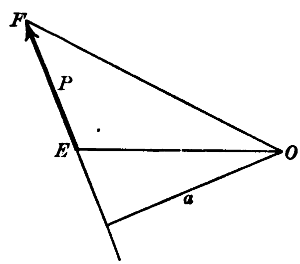


FIG. 25

The moment may be represented geometrically as follows: let EF represent the magnitude of P, drawn to the desired scale, and draw EO and FO. The area of the triangle $OEF = \frac{1}{2}\ EFa$, or $EFa = 2\ \triangle\ OEF$; that is, *the moment of the force with respect to a point is geometrically represented by twice the area of the triangle, whose base is the line representing the magnitude of the force and whose vertex is the given point.*

If the moment of the force is negative, then the moment is represented by minus twice the area of the triangle.

22. Varignon's Theorem of Moments. — *The moment of the resultant of two concurring forces with respect to any point in their plane is equal to the algebraic sum of the moments of the two forces with respect to the same point.*

Let P and Q be any two concurrent forces and O a point in their plane. Through O draw a line parallel to the line of action of one of the forces, as P. Let the segment AC cut off on the line of Q by this line be taken to

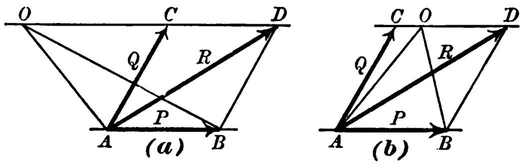


FIG. 26

represent the force Q, and let AB represent the force P to the same scale. Then, using the upper sign for case (a) and the lower for case (b), Fig. 26,

$$\text{mom } P + \text{mom } Q = 2\,(\text{area of } OAB \pm \text{area of } OAC).$$

$$\text{But area of } OAB = \text{area of } ABD = \text{area of } ACD.$$

$$\therefore 2\,(\text{area of } OAB \pm \text{area of } OAC) = 2 \text{ area of } OAD$$
$$= \text{mom } R.$$

$$\therefore \text{mom } P + \text{mom } Q = \text{mom } R.$$

23. Moment of a Force with Respect to a Line. — Let P be any force and AB any line (Fig. 27). The moment of P with respect to AB is defined as follows: Resolve P into two components one of which is parallel to, and the other perpendicular to, AB. The product of the component perpendicular to AB and the perpendicular distance of this component from AB is called the

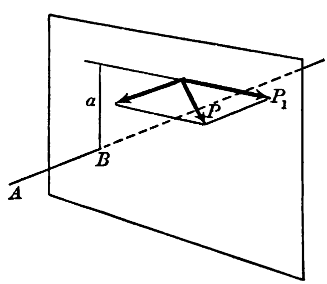


FIG. 27

moment of P with respect to AB. (In Fig. 27 the moment of P with respect to AB is aP_1.) More briefly the definition is: The moment of a force with respect to a line is the moment of the projection of the force upon a plane perpendicular to the line, with respect to the intersection of the line and the plane. The moment is considered positive or negative according as the tendency to rotation by the force is counter-clockwise or clockwise. The sign of the moment will then change if the observer changes from one side of the plane on which the projection is made to the other, and in comparing the moments of several forces with respect to a line the forces should all be projected upon the same plane and their projections viewed from the same side of that plane.

24. Moment of the Resultant of Two Concurrent Forces with Respect to a Line. — *The sum of the moments of two concurrent forces with respect to any line is equal to the moment of their resultant with respect to that line.*

Proof: Let P and Q be any two concurrent forces, R their resultant, and AB any line (Fig. 28). Through the intersection of P and Q pass a plane MN perpendicular to AB, cutting AB in O. Project P, Q, and R on MN, the projections being respectively P_1, Q_1, and R_1.

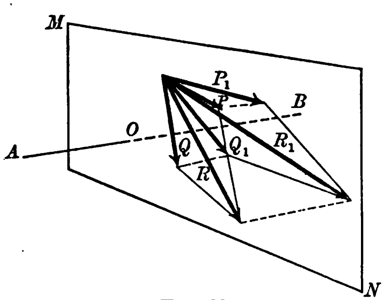


FIG. 28

From the definition of projection it follows at once that R_1 is the resultant of P_1 and Q_1. The moments of P, Q,

and R with respect to AB are respectively equal to the moments of P_1, Q_1, and R_1 with respect to O. (Definition.)

By Art. 22, with respect to O,

$$\text{mom } P_1 + \text{mom } Q_1 = \text{mom } R_1.$$

Therefore, with respect to AB,

$$\text{mom } P + \text{mom } Q = \text{mom } R.$$

If there are three or more forces, the application of the above theorem to the resultant of two of the forces and a third force, etc., proves that *the sum of the moments of any number of concurrent forces with respect to any line is equal to the moment of their resultant with respect to that line.*

COROLLARY. From the definition of the moment of a force with respect to a line it follows that the sum of the moments of two forces of equal numerical value but opposite in direction and acting in the same line is zero.

Use will be made of these principles in a later chapter.

CHAPTER III

PARALLEL FORCES

25. The Resultant of Two Parallel Forces. — In considering two parallel forces three cases arise: (*a*) when the forces are in the same direction; (*b*) when they are *unequal* and in opposite directions; (*c*) when they are *equal* and in opposite directions, but have different lines of action.

In case (*c*) the two forces form a *couple*. It will be shown later that there is no single force that will replace them.

In cases (*a*) and (*b*) the two forces have a resultant. Its value and line of action are found as follows:

Let the two forces be P_1 and P_2, acting at the points A_1 and A_2. At A_1 and A_2 in the line A_1A_2 put in two equal and opposite forces T_1 and T_2. These two forces will have no effect as far as the state of rest or motion of the body on which the forces act is concerned. Combine these

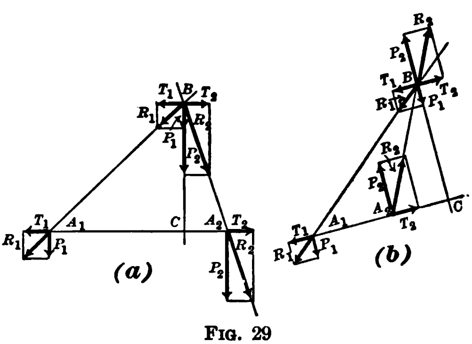


FIG. 29

forces with P_1 and P_2, respectively, obtaining the resultants R_1 and R_2. These resultants may be moved back to the point of intersection of their lines of action B, and resolved into components parallel to their original components. The two forces T_1 and T_2 then annul each other, and there are left the two forces P_1 and P_2 acting at B parallel to their original positions. The resultant then in case (a) is $R = P_1 + P_2$, and in case (b) $R = P_2 - P_1$.

Let the line of action of the resultant cut the line A_1A_2 in C, and let the distances from C to A_1 and A_2 be respectively d_1 and d_2.

Then from similar triangles, in either case,

$$\frac{BC}{d_1} = \frac{P_1}{T_1} \text{ and } \frac{BC}{d_2} = \frac{P_2}{T_2}.$$

By division, remembering that $T_1 = T_2$,

$$\frac{d_1}{d_2} = \frac{P_2}{P_1}.$$

Hence, *the resultant of two parallel forces acting in the same direction is equal to their sum, is parallel to the forces, and divides the line joining their points of application in the inverse ratio of the forces.*

The resultant of two unequal parallel forces acting in opposite directions is equal to their difference, acts in the direction of the larger force, and divides the line joining their points of application externally in the inverse ratio of the forces.

26. The Moment of the Resultant of Two Parallel Forces. —Applying the theorem of Art. 24 for the moment of

the resultant of concurrent forces to the forces of the preceding article,

$$\begin{aligned} \text{mom } R &= \text{mom } R_1 + \text{mom } R_2 \\ &= \text{mom } P_1 + \text{mom } T_1 + \text{mom } P_2 + \text{mom } T_2. \end{aligned}$$

But $\qquad \text{mom } T_1 + \text{mom } T_2 = 0$ (Art. 24, Cor.).

Therefore $\qquad \text{mom } R = \text{mom } P_1 + \text{mom } P_2.$

Or, *the moment of the resultant of two parallel forces with respect to any line is equal to the algebraic sum of the moments of the two forces with respect to that line.*

27. The Resultant of Any Number of Parallel Forces in Space. — By combining the resultant of two parallel forces with a third, and that resultant with another, and so on, the theorems of the two preceding articles may be extended at once to any number of parallel forces:

(1) *The resultant of any number of parallel forces in space is equal to their algebraic sum and acts parallel to the forces.*

(2) *The algebraic sum of the moments of any number of parallel forces in space with respect to any line is equal to the moment of their resultant with respect to that line.*

It follows at once that *if a set of parallel forces is in equilibrium, the sum of the moments of all the forces with respect to any line is zero,* since the resultant of the forces is zero, and hence the moment of the resultant is zero.

If, conversely, the sum of the moments of a set of parallel forces with respect to a line not parallel to the forces is zero, either the resultant of the forces must be zero, or else the line of action of the resultant must intersect the given line. If, however, the sum of the moments

of the forces about each of two lines not parallel to the forces which cannot intersect the same line parallel to the lines of action of the forces is zero, the resultant must be zero and the forces are in equilibrium.

If the lines of action of the forces all lie in one plane, the most convenient line about which to take moments is perpendicular to the plane of the forces.

The condition just stated for equilibrium in this case becomes: *If the sum of the moments of a set of parallel forces in one plane with respect to each of three points of*

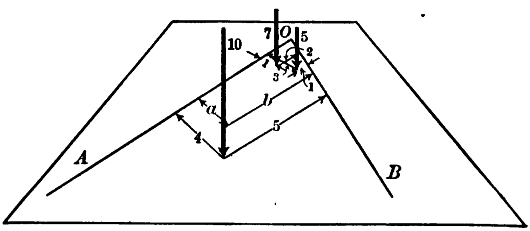


FIG. 30

the plane not in the same straight line is zero, the forces are in equilibrium.

Illustration. Forces of 7 lb., 5 lb., and 10 lb. act perpendicular to the plane of the lines OA and OB, as shown in Fig. 30. The distances of the lines of action of the forces from OA and OB are respectively 1, 3, 4 units and 2, 1, 5 units. To find where the line of action of the resultant cuts the plane OAB:

Let a and b be the distances of the line of action of the resultant from OA and OB respectively. The resultant is

$$R = 7 + 5 + 10 = 22 \text{ lb.},$$

acting in the direction of the forces.

Taking moments about the line OA,

$$a \cdot R = 1 \cdot 7 + 3 \cdot 5 + 4 \cdot 10,$$

or $$a = 2.82 \text{ units of space.}$$

Taking moments about OB,

$$b \cdot R = 2 \cdot 7 + 1 \cdot 5 + 5 \cdot 10,$$

or $$b = 3.14 \text{ units of space.}$$

Hence the resultant is equal to 22 lb., acts in the direction of the forces, and at distances of 2.82 and 3.14 units from OA and OB respectively.

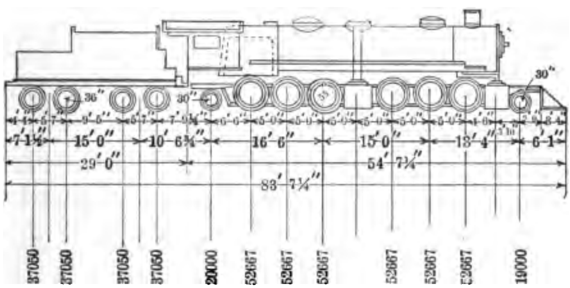


Fig. 31

Problem 18. Two parallel forces, one of 20 lb. and one of 100 lb., have lines of action 24 in. apart. Find the resultant in magnitude, direction, and line of action:

(1) When they are in the same direction.

(2) When they are in opposite directions.

Problem 19. A horizontal beam of length l is supported at its ends by two piers and loaded with a single load P at a distance of $\frac{l}{5}$ from one end. Find the reactions of the piers against the beam.

Problem 20. The locomotive shown in Fig. 31 is run upon a turntable whose length is 100 ft. Find the position of the engine so that the table will balance.

Problem 21. Weights of 100, 50, and 120 lb. are placed at the vertices A, B, and C respectively of an equilateral triangle 4 ft. on a side. Find the distances of the center of the forces from the lines AB and BC.

Problem 22. On a square table weights of 70, 80, and 100 lb. are placed as shown in Fig. 32. Find the position of a fourth weight of 90 lb. which will balance the given weights with respect to the two lines AB and CD.

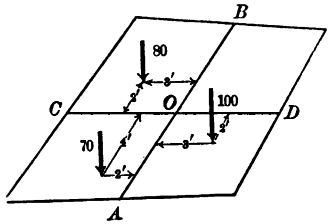


FIG. 32

Ans. $\frac{8}{9}$ ft. from AB, $3\frac{5}{9}$ ft. from CD in quadrant BOD.

28. Center of Parallel Forces. — In Art. 25 it was shown that the resultant of two parallel forces P_1 and P_2, acting at the points A_1 and A_2 respectively, divides the line A_1A_2 in the inverse ratio of the forces. If, then, without changing the points of application of the parallel forces

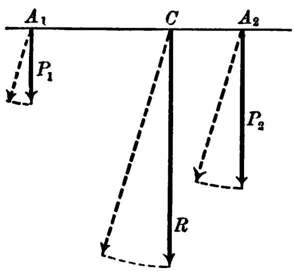


FIG. 33

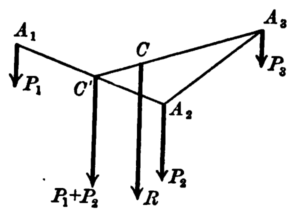


FIG. 34

or their magnitudes their direction be changed, the point on A_1A_2 through which the resultant passes is not changed, since it must still divide A_1A_2 in the ratio $P_2 : P_1$.

In the same way, if three parallel forces of fixed value P_1, P_2, P_3 act at three fixed points A_1, A_2, A_3, and if C'

is the point on A_1A_2 through which the resultant of P_1 and P_2 passes, then the point C on $C'A_3$ which divides $C'A_3$ in the ratio $P_3:(P_1+P_2)$ is a point through which the resultant of the three forces always passes no matter what their direction. The same reasoning applies to any number of parallel forces acting at definite points. For any set of parallel forces acting at definite points there is therefore a point through which the resultant always passes no matter what direction the parallel forces may take. This point is called the *center of the parallel forces.*

29. Graphical Construction for the Center of Parallel Forces. — Let the parallel forces P_1 and P_2 act at A_1 and

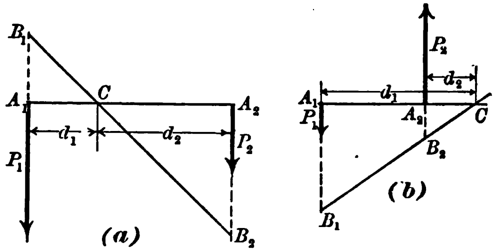


FIG. 35

A_2 (Fig. 35). From A_1 lay off A_1B_1 equal to P_2 but in the opposite direction to P_2. From A_2 lay off A_2B_2 equal to P_1 and in the same direction as P_1. The intersection of B_1B_2 and A_1A_2 is then the center of the two forces. For, letting $A_1C = d_1$ and $CA_2 = d_2$, we have, by similar triangles A_1B_1C and A_2B_2C,

$$\frac{d_1}{d_2} = \frac{A_1B_1}{A_2B_2}, \quad \text{or} \quad \frac{d_1}{d_2} = \frac{P_2}{P_1}.$$

Hence C is the point found in Art. 25 to be the center of the forces.

If there are more than two parallel forces, the continued application of the construction will locate the center of the forces.

Problem 23. Weights of 20, 45, and 70 lb. are suspended from points on a straight line at distances of 2, 5, and 7 ft. respectively from a point O. Find by graphical construction the center of the forces exerted by the weights. Check by the theory of moments.

Problem 24. Locate graphically the center of the three forces in the illustration of Art. 27. (N.B. The forces may be assumed to act in the plane OAB without changing the position of the center.)

30. In Regard to Signs. — Given a set of parallel forces, P_1, P_2, P_3, etc., acting at definite points.

Let OY be a line in a plane perpendicular to the lines of action of the forces. For convenience assume the lines of action of the forces to be vertical. Let it be agreed that forces acting upward shall be positive and those acting downward negative.

Let x_1, x_2, x_3, $\cdots$ be the perpendiculars from OY to the lines of the forces P_1, P_2, P_3, $\cdots$, x being counted positive if measured from the axis OY toward the right and negative if measured from OY toward the left. Then, when observed in the direction YO, Px is the moment of P with respect to OY, not only in numerical value but in sign. For if both P and x are positive, as P_1 and x_1, or both negative, as P_3 and x_3, the product is positive. Inspection of the figure shows that the tendency to rotation in both cases is counterclockwise, which has been defined as positive.

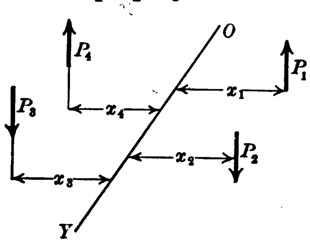


FIG. 36

If x is positive and P is negative, as x_2 and P_2, or if x is negative and P is positive, as x_4 and P_4, the product is negative, and an inspection of the figure shows that the tendency to rotation is clockwise, which has been defined as negative.

The sum of the moments of the forces with respect to OY is then $P_1x_1 + P_2x_2 + P_3x_3 + P_4x_4 + \cdots$. This sum is denoted by ΣPx.

31. Coördinates of the Center of a System of Parallel Forces.—Let P_1, P_2, P_3, $\cdots$ be a set of parallel forces whose points of application are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), $\cdots$ in rectangular coördinates. Denote the center of the forces by $(\bar{x}, \bar{y}, \bar{z})$. As was shown in Art. 28 the position of the center of the forces is independent of the direction in which they act. Assume then that they act parallel to OZ.

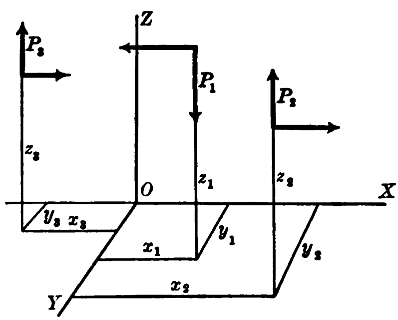


FIG. 37

Taking moments about OY, we have the moment of the resultant equal to the sum of the moments of the forces, or

$$\bar{x}(P_1 + P_2 + P_3 + \cdots) = P_1x_1 + P_2x_2 + P_3x_3 + \cdots$$

or

$$\bar{x}\Sigma P = \Sigma Px.$$

Taking moments about OX,

$$\bar{y}\Sigma P = \Sigma Py.$$

Assuming the forces to act parallel to OX and taking moments about OY, we obtain

$$\bar{z}\Sigma P = \Sigma Pz.$$

Therefore

$$\bar{x} = \frac{\Sigma Px}{\Sigma P},$$

$$\bar{y} = \frac{\Sigma Py}{\Sigma P},$$

$$\bar{z} = \frac{\Sigma Pz}{\Sigma P}.$$

Problem 25. Forces of 40, 65, and 70 lb. act in the same direction from the points (4, 3, 1), (5, −2, 3), and (2, 3, 6), respectively. Find the resultant and the center of the forces.

Problem 26. Three equal parallel forces act at the vertices of a triangle in the same direction; prove that their resultant acts at the intersection of the medians of the triangle. Solve by applying the theorem of moments, and check by graphical construction.

Problem 27. Four equal forces act in the same direction at the vertices of a regular tetrahedron. Find the center of the forces by taking moments. Also locate the center graphically.

32. Arrangement of the Work for Computation. — If several parallel forces and their points of application are given, it is sometimes worth while to tabulate the work in some form such as the following:

Forces	Coördinates			Moments		
	x	y	z	Px	Py	Pz
P_1	x_1	y_1	z_1	P_1x_1	P_1y_1	P_1z_1
P_2	x_2	y_2	z_2	P_2x_2	P_2y_2	P_2z_2
P_3	x_3	y_3	z_3	P_3x_3	P_3y_3	P_3z_3
P_4	x_4	y_4	z_4	P_4x_4	P_4y_4	P_4z_4
Resultant	$\bar{x}$	$\bar{y}$	$\bar{z}$	ΣPx	ΣPy	ΣPz

The values of quantities in the columns one, two, three, and four are given except in the last row. The values of the quantities in columns five, six, and seven are then computed and their sums and the sum of column one placed at the foot. The values of $\bar{x}$, $\bar{y}$, and $\bar{z}$ are then easily computed.

Problem 28. Using the above arrangement, find the coördinates of the center of the following parallel forces: $P_1 = 50$ lb., $P_2 = 100$ lb., $P_3 = 300$ lb., $P_4 = 10$ lb., and $P_5 = -400$ lb., the points of application being respectively $(2, 1, -5)$, $(-1, -2, 4)$, $(2, 1, -2)$, $(-2, 1, 1)$, $(1, 1, 1)$. *Ans.* $(3, -4, -14)$.

Problem 29. Parallel forces P_1, P_2, P_3, and P_4 act at the corners of a rectangle 3 ft. by 2 ft. and perpendicular to its plane. Find the point of application of the resultant, if $P_1 = 10$ lb., $P_2 = 50$ lb., $P_3 = 100$ lb., $P_4 = 200$ lb., P_1 and P_2 being 2 ft. apart, and P_3 on same side as P_2.

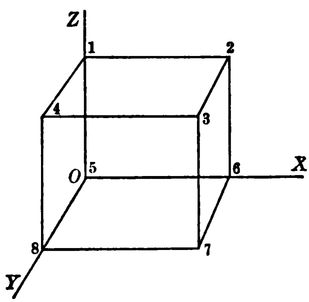


FIG. 38

Problem 30. Eight parallel forces act at the corners of a one-inch cube. Find the point of application of the resultant force, if $P_1 = 30$ lb., $P_2 = 50$ lb., $P_3 = 10$ lb., $P_4 = 20$ lb., $P_5 = 100$ lb., $P_6 = 5$ lb., $P_7 = 10$ lb., $P_8 = 40$ lb. The subscripts of the forces acting at the various vertices are shown in the figure (Fig. 38).

Ans. (.283, .302, .415).

CHAPTER IV

CENTER OF GRAVITY

33. Definition of the Center of Gravity. — The center of gravity of a body may be defined as the point of application of the resultant attraction of the earth for that body, and the center of gravity of several bodies considered together, as the point of application of the resultant attraction of the earth for the bodies. The attention of the student is called to the fact that the forces acting upon the particles of a body, due to the attraction of the earth, are not parallel, but meet in the center of the earth. For all practical purposes, however, they are considered parallel.

The center of gravity of many simple bodies can be found by inspection. For example, for a sphere of uniform material it is evident that the line of action of the earth's attraction always passes through the center. For a rectangular block of uniform material the same is true.

If a body be divided up into a number of parts whose centers of gravity are known, the whole weight of the body is the resultant of the known weights of the parts acting at known points, and hence the theorem of moments may be applied to determine the position of the center of gravity of the body; *i.e.* the equations

$$\bar{x} = \frac{\Sigma xP}{\Sigma P}, \quad \bar{y} = \frac{\Sigma yP}{\Sigma P}, \quad \bar{z} = \frac{\Sigma zP}{\Sigma P}$$

may be used where P_1, P_2, etc., represent the weights of the known parts, x_1, x_2, etc., the abscissas of the centers of gravity of these parts, etc.

As an illustration consider the problem of finding the center of gravity of the solid shown in Fig. 39.

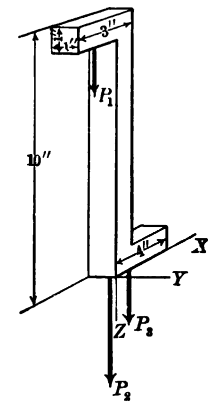


FIG. 39

The figure represents a Z-iron of the same cross section throughout, and P_1, P_2, and P_3 are the weights of the individual parts (considering the Z-iron as divided into three parts — two legs and the connecting vertical portion). If the weight of a cubic inch of iron $= .26$ lb., $P_1 = .78$ lb., $P_2 = 2.08$ lb., $P_3 = 1.04$ lb., and therefore $R = 3.9$ lb. The points of application of P_1, P_2, and P_3 are $(-\frac{1}{2}, -\frac{1}{2}, 9\frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2}, 5)$, and $(2, -\frac{1}{2}, \frac{1}{2})$, respectively, so that

$$\bar{x} = \frac{.78(-\frac{1}{2}) + 2.08(\frac{1}{2}) + 1.04(2)}{3.9} = .70 \text{ in.},$$

$$\bar{y} = \frac{.78(-\frac{1}{2}) + 2.08(-\frac{1}{2}) + 1.04(-\frac{1}{2})}{3.9} = -.50 \text{ in.},$$

$$\bar{z} = \frac{.78(\frac{19}{2}) + 2.08(5) + 1.04(\frac{1}{2})}{3.9} = 4.7 \text{ in.}$$

This point $\bar{x}$, $\bar{y}$, $\bar{z}$ is, in this case, the center of gravity of the Z-iron.

34. Centers of Gravity of Uniform Bodies and of Areas. — If P_1, P_2, P_3, etc., are the weights of the parts of a body or of several bodies, whose unit weights are, respectively, γ_1, γ_2, γ_3, etc., and volumes V_1, V_2, V_3, etc., we may

write $\gamma_1 V_1$ for P_1, $\gamma_2 V_2$ for P_2, $\gamma_3 V_3$ for P_3, etc. The formulæ for $\bar{x}$, $\bar{y}$, $\bar{z}$ then become

$$\bar{x} = \frac{\gamma_1 V_1 x_1 + \gamma_2 V_2 x_2 + \gamma_3 V_3 x_3 + \text{etc.}}{\gamma_1 V_1 + \gamma_2 V_2 + \gamma_3 V_3 + \text{etc.}} = \frac{\Sigma \gamma V x}{\Sigma \gamma V},$$

$$\bar{y} = \frac{\Sigma \gamma V y}{\Sigma \gamma V}, \quad \bar{z} = \frac{\Sigma \gamma V z}{\Sigma \gamma V}.$$

And if the bodies are all of the same material and so have the same heaviness, γ is constant and may be taken outside the summation sign, where it cancels out. This gives values for $\bar{x}$, $\bar{y}$, and $\bar{z}$,

$$\bar{x} = \frac{\Sigma V x}{\Sigma V}, \quad \bar{y} = \frac{\Sigma V y}{\Sigma V}, \quad \bar{z} = \frac{\Sigma V z}{\Sigma V},$$

formulæ exactly similar to those of Art. 33, where the P's are replaced by V's.

If the bodies are thin plates of the same material, of constant thickness b, we may write for V_1, V_2, V_3, etc. bF_1, bF_2, bF_3, etc., where the F's represent the areas of the faces of the plates. Making this substitution for the V's, $\bar{x}$, $\bar{y}$, $\bar{z}$ may be written

$$\bar{x} = \frac{bF_1 x_1 + bF_2 x_2 + bF_3 x_3 + \text{etc.}}{bF_1 + bF_2 + bF_3 + \text{etc.}} = \frac{\Sigma F x}{\Sigma F},$$

$$\bar{y} = \frac{\Sigma F y}{\Sigma F}, \quad \bar{z} = \frac{\Sigma F z}{\Sigma F};$$

the b, being a constant factor, cancels out. These formulæ are taken as defining the "center of gravity of an area" and are much used by engineers for finding the center of gravity of sections of angles, channels, T-sections, Z-sections, etc. Usually, the xy-plane is taken in the plane of the area and only the values of $\bar{x}$ and $\bar{y}$ are needed.

Problem 31. Find the center of gravity of the channel section shown in Fig. 40.

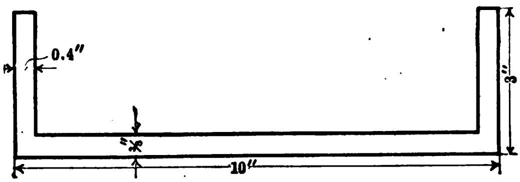


FIG. 40

Problem 32. Find the center of gravity of the T-section shown in Fig. 41.

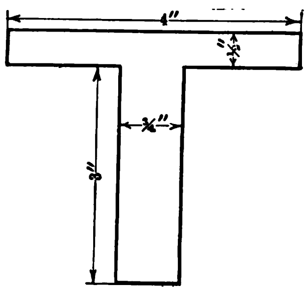


FIG. 41

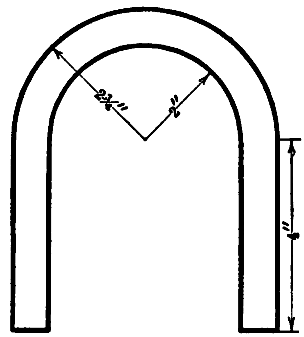


FIG. 42

Problem 33. Find the center of gravity of the U-section shown in Fig. 42. Given the fact that the center of gravity of a semicircular area is $\frac{4r}{3\pi}$ from the diameter. (See Prob. 41.)

Problem 34. Find the position of the center of gravity of a trapezoidal area, the lengths of whose parallel sides are a_1 and a_2, respectively, and the distance between them h. (See Fig. 43.)

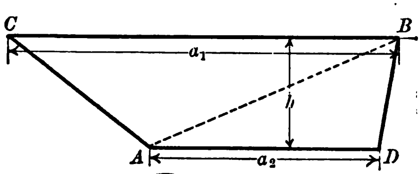


FIG. 43

HINT. Draw the diagonal AB and call the triangle ACB, F_1, and the triangle ABD, F_2. Given, the center of gravity of a triangle is $\frac{1}{3}$ the distance from the base to the vertex. (See Prob. 39.) Select AD as the x-axis, then

$$\bar{y} = \frac{F_1 y_1 + F_2 y_2}{F_1 + F_2},$$

where $y_1 = \frac{2}{3} h$ and $y_2 = \frac{1}{3} h$. The center of gravity is seen to lie on a line joining the middle points of the parallel sides.

Problem 35. A cylindrical piece of cast iron, whose height is 6 in. and the radius of whose base is 2 in., has a cylindrical hole of 1 in. radius drilled in one end, the axis of which coincides with the axis of the cylinder. The hole was originally 3 in. deep, but has been filled with lead until it is only 1 in. deep. Find the center of gravity of the body, the unit weight of lead being 710 and of cast iron 450 (Fig. 44).

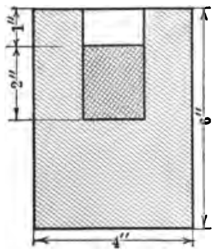


FIG. 44

Problem 36. Find the center of gravity of a portion of a reinforced concrete beam. (See Fig. 45.) The beam is reinforced with three half-inch steel rods, centers 1 in. from the bottom of the beam and 1

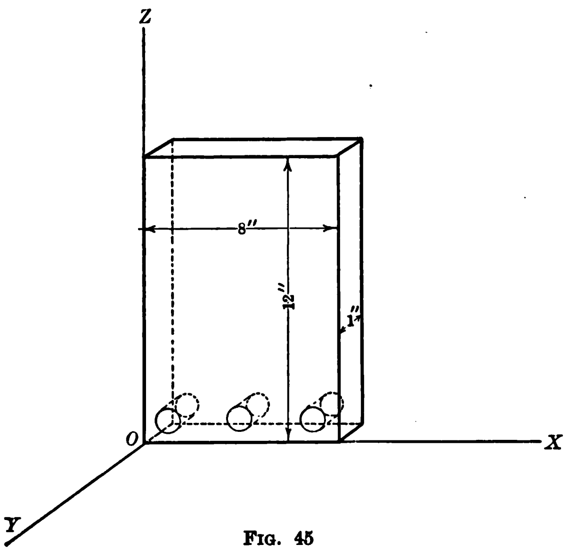


FIG. 45

in. from the sides. The center of the middle rod is 4 in. from the sides.

(γ for steel = 490 lb. per cubic foot;
γ for concrete = 125 lb. per cubic foot.)

NOTE. It is seen that the thickness cancels out of the expression for the center of gravity, and might, therefore, have been neglected.

35. Center of Gravity of a Body with a Portion Removed. — It is sometimes convenient in finding the center of gravity of a body to regard it as a larger body from which a portion has been removed. The weight of the given body can then be regarded as the resultant of the weight of the larger body acting at its center of gravity and a force equal to the weight of the portion removed acting at its center of gravity in the opposite direction.

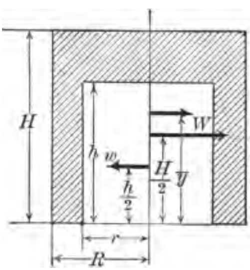


FIG. 46

To illustrate this consider the cylinder of Fig. 46 of height H and radius R from which a cylindrical portion of height h and radius r and having the same axis is removed from one end.

If W is the weight of the whole cylinder, and w the weight of the portion removed, then the center of gravity of the remaining body is the center of the two forces W and w acting as shown in the figure. If γ is the heaviness,

$$W = \gamma \pi R^2 H,$$
$$w = \gamma \pi r^2 h.$$

The distance $\bar{y}$ from the open end to the center of gravity of the body is therefore

$$\bar{y} = \frac{\gamma\pi R^2 H \frac{H}{2} - \gamma\pi r^2 h \frac{h}{2}}{\gamma\pi R^2 H - \gamma\pi r^2 h} = \frac{1}{2} \frac{R^2 H^2 - r^2 h^2}{R^2 H - r^2 h}.$$

In general the formula is,

$$\bar{x} = \frac{x_1 W_1 - x_2 W_2}{W_1 - W_2},$$

where W_2 is the weight removed from the weight W_1 and x_1 and x_2 are the abscissas of the centers of gravity of the weights W_1 and W_2, respectively, before the removal of the weight W_2.

Problem 37. Through a circular disk 1 ft. in diameter a hole 4″ in diameter is bored with its axis distant 3″ from the axis of the disk. Find the position of the center of gravity of the remainder.

Problem 38. From two adjacent corners of a cube of edge 1 ft. cubes of 2-inch and 3-inch edges are removed. Find the center of gravity of the remaining portion.

36. Center of Gravity Determined by Integration. — In many cases of areas and solids the position of the center of gravity may be determined by integration. The method as applied to areas is as follows:

Let F be any area in the xy-plane. In order to deal with actual forces think of F as the face of a thin sheet of uniform thickness and uniform material and let γ be the weight of the sheet per unit area in the surface F. Divide the area up into elements of area ΔF. (This may be done in a variety of ways, Fig. 47.) The weight of such an element is then $\gamma\Delta F$. The weight of the sheet may then be replaced by a set of parallel forces of magnitude $\gamma\Delta F$ acting at the centers of these elementary areas.

The centers of the elementary areas and the elementary areas themselves may be determined approximately, and if x' is the abscissa of the approximate center* of ΔF,

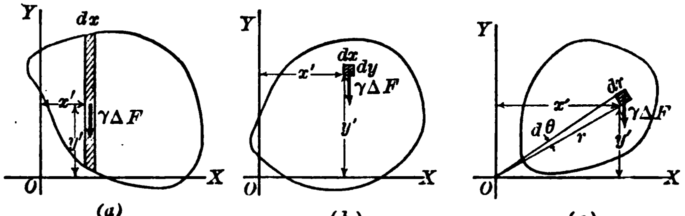


FIG. 47

then an approximate value for the abscissa of the center of the parallel forces is

$$\bar{x}' = \frac{\Sigma x' \gamma \Delta F}{\Sigma \gamma \Delta F}. \qquad \text{(Art. 33)}$$

The limiting value of $\bar{x}'$ as the elementary areas are indefinitely decreased is defined as the abscissa, $\bar{x}$, of the center of gravity of the area F. Hence,

$$\bar{x} = \frac{\operatorname{Lim} \Sigma x' \gamma \Delta F}{\operatorname{Lim} \Sigma \gamma \Delta F}$$

as ΔF approaches the limit zero. By a theorem of calculus this may be written

$$\bar{x} = \frac{\int x' \gamma dF}{\int \gamma dF},$$

* The approximate center must of course be such that the approximate center and the true center of gravity of the element have the same limiting position as the elementary area is indefinitely decreased. If both dimensions of ΔF approach the limit zero as in (b) and (c), Fig. 47, the approximate center may be taken as any point of ΔF.

or, since γ is constant,

$$\bar{x} = \frac{\int x' dF}{\int dF},$$

the integration being taken to include the whole area F. Similarly

$$\bar{y} = \frac{\int y' dF}{\int dF}.$$

In deriving the latter formula the forces would be thought of as acting parallel to the x-axis.

In the use of these formulæ it must be kept in mind that x' and y' are the coördinates of the approximate center of the elementary area ΔF.

Illustration. To find the center of gravity of the area bounded by the x-axis, the curve $y = \sin x$, and the line $x = \frac{\pi}{2}$.

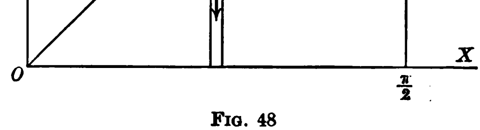


FIG. 48

Divide the area up into strips parallel to the y-axis of width Δx. Then considering any strip whose middle ordinate intersects the curve in the point (x, y), the value of ΔF is $y\Delta x$ approximately, the value of x' is x, and the value of y' is $\frac{y}{2}$ (Fig. 48).

Then
$$\bar{x} = \frac{\int_0^{\frac{\pi}{2}} xy\,dx}{\int_0^{\frac{\pi}{2}} y\,dx} = \frac{\int_0^{\frac{\pi}{2}} x \sin x\,dx}{\int_0^{\frac{\pi}{2}} \sin x\,dx}$$

$$= \frac{(-x\cos x + \sin x)\Big]_0^{\frac{\pi}{2}}}{-\cos x\Big]_0^{\frac{\pi}{2}}} = \frac{1}{1} = 1,$$

$$\bar{y} = \frac{\int_0^{\frac{\pi}{2}} \frac{y}{2}\,y\,dx}{\int_0^{\frac{\pi}{2}} y\,dx} = \frac{\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2 x\,dx}{\int_0^{\frac{\pi}{2}} \sin x\,dx}$$

$$= \frac{(\frac{1}{4}x - \frac{1}{8}\sin 2x)\Big]_0^{\frac{\pi}{2}}}{-\cos x\Big]_0^{\frac{\pi}{2}}} = \frac{\frac{\pi}{8}}{1} = \frac{\pi}{8}.$$

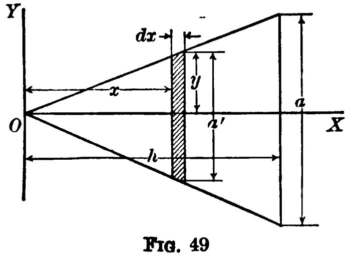


FIG. 49

Problem 39. Find the center of gravity of a triangle whose altitude is h and whose base is a. Take the origin at the vertex and draw the x-axis perpendicular to the base. (See Fig. 49.)

$$\bar{x} = \frac{\int x' dF}{\int dF}.$$ Here $dF = a'dx$, and from similar triangles

$$a' = \frac{a}{h}x, \text{ so that } dF = \frac{a}{h}xdx,$$

and

$$\bar{x} = \frac{\frac{a}{h}\int_0^h x^2 dx}{\frac{a}{h}\int_0^h xdx} = \frac{\left.\frac{x^3}{3}\right]_0^h}{\left.\frac{x^2}{2}\right]_0^h} = \frac{2}{3}h.$$

The center of gravity is $\frac{2}{3}$ the distance from the vertex to the base, and since the *median* is a line of symmetry, it is a point on the median. It is, in fact, the point where the medians of the triangle intersect.

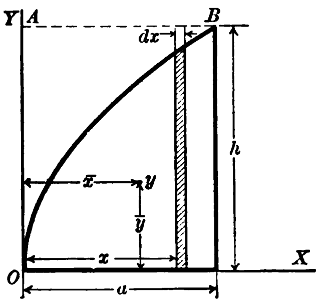


FIG. 50

Problem 40. Find the center of gravity of a parabolic area shown in Fig. 50, the equation of the parabola being $y^2 = 2px$.

Here $dF = ydx$, so that

$$\bar{x} = \frac{\int_0^a xydx}{\int_0^a ydx} = \frac{\sqrt{2p}\int_0^a x^{\frac{3}{2}}dx}{\sqrt{2p}\int_0^a x^{\frac{1}{2}}dx} = \frac{\left.\frac{2}{5}x^{\frac{5}{2}}\right]_0^a}{\left.\frac{2}{3}x^{\frac{3}{2}}\right]_0^a} = \frac{3}{5}a.$$

It is left as a problem for the student to show that $\bar{y} = \frac{3}{8}h$.

Problem 41. Find the center of gravity of a sector of a flat ring outside radius R_1 and inside radius R_2. (See Fig. 51.) Let the

angle of the sector be $2\,\theta$. Take the origin at the center and let the x-axis bisect the angle $2\,\theta$.

Here $$dF = \rho d\rho d\alpha, \text{ and } x = \rho \cos \alpha, \text{ so that}$$

$$\bar{x} = \frac{\int\int \cos \alpha d\alpha \cdot \rho^2 d\rho}{\int\int \rho d\rho d\alpha}.$$

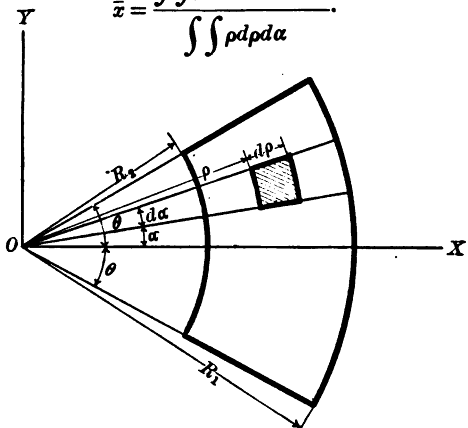


FIG. 51

Integrating the numerator first,

$$\int_{-\theta}^{+\theta} \cos \alpha d\alpha \int_{R_2}^{R_1} \rho^2 d\rho = \frac{R_1^3 - R_2^3}{3} \int_{-\theta}^{+\theta} \cos \alpha d\alpha = \frac{R_1^3 - R_2^3}{3} (2 \sin \theta).$$

Integrating the denominator,

$$\int_{-\theta}^{+\theta} \alpha \int_{R_2}^{R_1} \rho d\rho = \frac{R_1^2 - R_2^2}{2} \int_{-\theta}^{+\theta} d\alpha = (R_1^2 - R_2^2)\theta.$$

Therefore, $$\bar{x} = \frac{2}{3} \frac{R_1^3 - R_2^3}{R_1^2 - R_2^2} \frac{\sin \theta}{\theta}.$$

If $R_2 = 0$, the sector becomes the sector of a circle, and $\bar{x}$ becomes

$$\bar{x} = \frac{2}{3} R_1 \frac{\sin \theta}{\theta}.$$

If the sector is a semicircle, that is, if $2\,\theta = \pi$, then, since $\theta = \frac{\pi}{2}$,

$$\bar{x} = \frac{4\,R_1}{3\,\pi}.$$

Problem 42. Find the center of gravity of a semi-ellipse (Fig. 52) whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

$$dF = 2\,ydx = 2\frac{b}{a}\sqrt{a^2-x^2}\cdot dx,$$

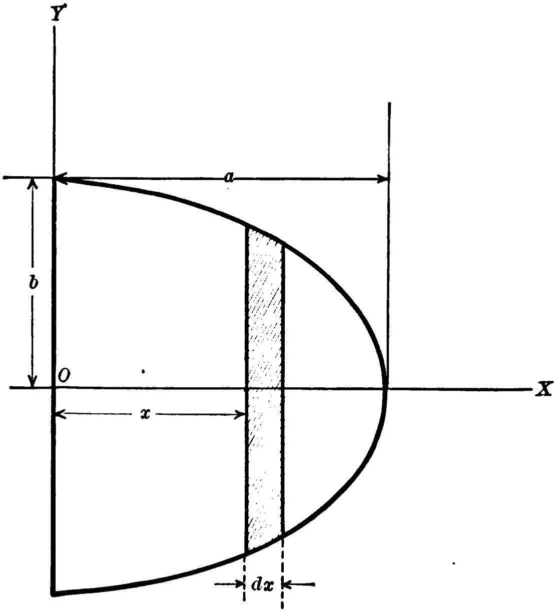


FIG. 52

therefore

$$\bar{x}=\frac{2\dfrac{b}{a}\displaystyle\int_0^a x\sqrt{a^2-x^2}\,dx}{2\dfrac{b}{a}\displaystyle\int_0^a\sqrt{a^2-x^2}\,dx}=\frac{-\dfrac{1}{3}(a^2-x^2)^{\frac{3}{2}}\Big]_0^a}{\dfrac{1}{2}\Big[x\sqrt{a^2-x^2}+a^2\sin^{-1}\dfrac{x}{a}\Big]_0^a}$$

$$=\frac{\dfrac{a^3}{3}}{\dfrac{a^2}{2}\cdot\dfrac{\pi}{2}}=\frac{\mathbf{4\,a}}{\mathbf{3\,\pi}}.$$

Problem 43. Find the center of gravity of the area between the parabola, the y-axis, and the line AB in Problem 40.

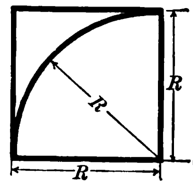


FIG. 53

Problem 44. A quadrant of a circle is taken from a square whose sides equal the radius of the circle. (See Fig. 53.) Find the center of gravity of the remaining area.

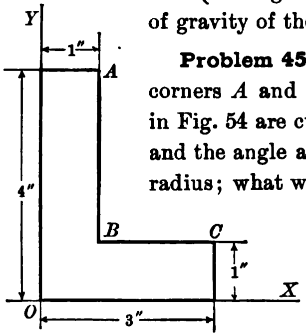


FIG. 54

Problem 45. Suppose that the corners A and C of the angle iron in Fig. 54 are cut to the arc of a circle of $\frac{5}{16}$ in. radius and the angle at B is filled to the arc of a circle $\frac{3}{8}$ in. radius; what would be the change in $\bar{x}$ and $\bar{y}$?

37. Center of Gravity of a Solid. — By dividing a solid into infinitesimal parts and taking the moments of the weights of these parts with respect to the three coördinate axes, the coördinates of the center of gravity of the solid may be found.

As an illustration, suppose it is desired to obtain the center of gravity of a right circular cone of altitude h and radius of base r. Take the x-axis as the axis of the cone with the vertex at the origin. (See Fig. 55.) It is evident that $\bar{y} = 0$ and $\bar{z} = 0$, so that it is only necessary to find $\bar{x}$. The

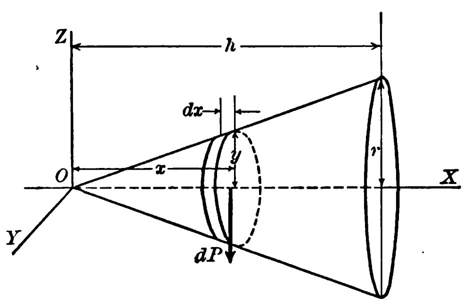


FIG. 55

volume, dv, cut from the cone by two parallel planes, perpendicular to OX and separated by a distance dx, is $\pi y^2 dx$, and the weight of this dv is $\gamma\pi y^2 dx = dP$. Therefore

$$\bar{x} = \frac{\int x' dP}{\int dP} = \frac{\int_0^h x\gamma\pi y^2 dx}{\int_0^h \gamma\pi y^2 dx}.$$

But from similar triangles $y : x :: r : h$ or $y = \frac{r}{h}x$. This gives

$$\bar{x} = \frac{\gamma\pi \frac{r^2}{h^2}\int_0^h x^3 dx}{\gamma\pi \frac{r^2}{h^2}\int_0^h x^2 dx} = \frac{\left.\frac{x^4}{4}\right]_0^h}{\left.\frac{x^3}{3}\right]_0^h} = \frac{3}{4}h.$$

The expressions for $\bar{x}$, $\bar{y}$, and $\bar{z}$, involving dP, may be changed to similar ones involving dv, and these become for homogeneous bodies, since $dP = \gamma dv$,

$$\bar{x} = \frac{\int x' dv}{\int dv}, \quad \bar{y} = \frac{\int y' dv}{\int ds}, \quad \bar{z} = \frac{\int z' dv}{\int dv}.$$

The center of gravity of thin homogeneous wires of constant cross section may be found by replacing the dv in the above formulæ by ads, where a is the constant area of cross section and ds is a distance along the curve. The formulæ then become

$$\bar{x} = \frac{\int x' ds}{\int ds}, \quad \bar{y} = \frac{\int y' ds}{\int ds}, \quad \bar{z} = \frac{\int z' ds}{\int ds}.$$

Problem 46. Find the center of gravity of a hemisphere, the radius of the sphere being r. Let the equation of the generating circle of the surface be $x^2 + y^2 = r^2$. Divide the hemisphere into

slices of thickness dx by planes parallel to the plane face of the hemisphere.

Then $$\bar{x} = \frac{\int x' dP}{\int dP}, \text{ where } dP = \gamma\pi y^2 dx = \gamma\pi(r^2 - x^2)dx.$$

Show that $$\bar{x} = \frac{3}{8} r.$$

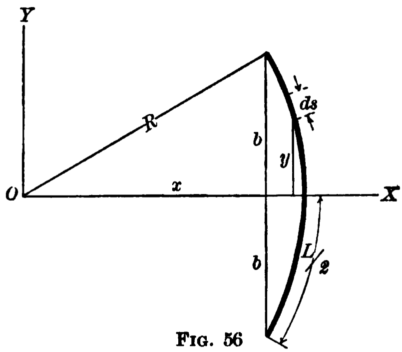


FIG. 56

Problem 47. Find the center of gravity of a portion of circular wire (Fig. 56) of length L and whose chord $= 2b$. Take the center of the circular arc as origin and let the x-axis bisect L. Then

$$\bar{x} = \frac{\int x ds}{\int ds}.$$

But $x^2 + y^2 = R^2$.

$\therefore 2xdx + 2ydy = 0.$

$$\therefore ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{y}{x}\right)^2 + 1} \cdot dy$$

$$= \frac{R}{x} dy.$$

$$\therefore \bar{x} = \frac{R\int_{-b}^{+b} dy}{L} = \frac{2Rb}{L} = \frac{\textbf{radius} \times \textbf{chord}}{\textbf{arc}}.$$

For a semicircular wire

$$\bar{x} = \frac{\text{Diameter}}{\pi}.$$

Problem 48. Find the center of gravity of a paraboloid of revolution. If the equation of the generating curve is $y^2 = 2px$, and the greatest value of x is a, show that

$$\bar{x} = \frac{2}{3}a.$$

SUGGESTION. Use the same method as that used for the right circular cone.

Problem 49. (*a*) Show that the center of gravity of the circular sector AOB (Fig. 57) of angle 2α and chord $2d$ is given by

$$\bar{x} = \frac{2}{3}\frac{d}{\alpha} = \frac{2}{3}\frac{\text{radius} \times \text{chord}}{\text{arc}}.$$

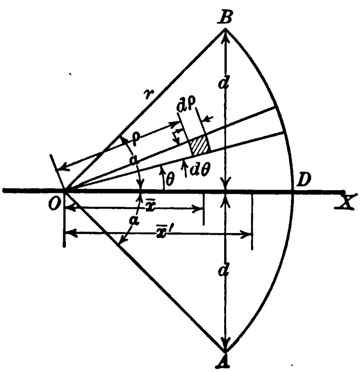


FIG. 57

(*b*) From this result and the known position of the center of gravity of the triangle, prove that the center of gravity of the segment ADB is given by

$$\bar{x}' = \frac{2d^3}{3F}$$

where F is the area of the segment.

SUGGESTION. In (*a*) use polar coördinates. The element of area is $\rho d\theta d\rho$ and

$$\bar{x} = \frac{\int_0^a \int_0^r \rho^2 \cos\theta d\theta d\rho}{\int_0^a \int_0^r \rho d\theta d\rho}.$$

Problem 50. Show that the center of gravity of any pyramid is in a plane parallel to the base which cuts the altitude at $\frac{3}{4}$ the distance from vertex to base.

Problem 51. Show that the center of gravity of a hemispherical surface bisects the radius perpendicular to the plane of the base of the surface.

Problem 52. Show that the center of gravity of a spherical zone is midway between the planes of the bases of the zone.

Problem 53. Show that the center of gravity of the surface of a right circular cone is at $\frac{2}{3}$ the distance from the vertex to the base.

Problem 54. Show that the center of gravity of the frustum of a right circular cone, the radii of the bases being R and r and the altitude H, is distant

$$\frac{H(R^2 + 2\,Rr + 3\,r^2)}{4(R^2 + Rr + r^2)}$$

from the base of radius R. Find also the distance of the center of gravity from the base of radius r.

Problem 55. A casting is in the form of a hollow cylinder with one end closed. The thickness of the end is $\frac{3}{4}$ in., the length of the casting is 12 in., and the radii of the inner and outer surfaces of the cylinder are $5\frac{1}{2}$ and 6 in. Find the position of the center of gravity of the casting.

Problem 56. If the above casting is filled with material $\frac{1}{5}$ as heavy as the material of the casting, find the position of the center of gravity.

Problem 57. A hemispherical shell of inner and outer radii r and R rests upon a hollow cylinder of the same material of height H

and inner and outer radii r and R. Find the distance of the center of gravity from the base of the cylinder.

Problem 58. From a right circular cone of height 20 in. and radius of base 10 in. a cylinder of height 10 in. and radius of base 3 in. is removed, the base and axis of the cylinder being in the base and axis of the cone. Find the distance from the base of the cone to the center of gravity of the part remaining.

Problem 59. If the cylindrical portion of the preceding problem is filled with material n times as heavy as the material of the cone, find the distance from the base to the center of gravity.

38. Center of Gravity of Counterbalance of Locomotive Drive Wheel. — In Fig. 58 the drive wheel is indicated by the circle and the counterbalance by the portion inclosed by the heavy lines, the point O is the center of the wheel, and α is the angle subtended by the counterbalance. The

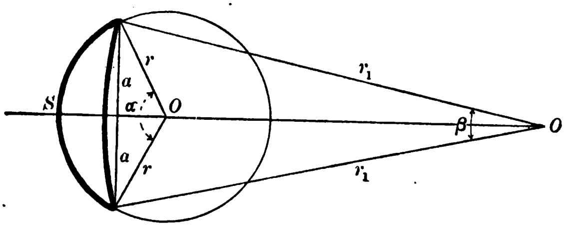


FIG. 58

point O' is the center of the circle forming the inner boundary of the counterbalance, and β is the angle subtended by the counterbalance at this point. Let F_1 represent the area of the segment of radius r and F_2 the area of the segment of radius r_1. Also let x_1 represent the distance of the center of gravity of F_1 from O, and x_2' the

distance of the center of gravity of F_2 from O'. Then, from Problem 49,

$$x_1 = \frac{2\,a^3}{3\,F_1} \text{ and } x_2' = \frac{2\,a^3}{3\,F_2}.$$

But x_2, the distance of the center of gravity of F_2 from O,

$$= x_2' - OO' = \frac{2\,a^3}{3\,F_2} - \left(r_1 \cos\frac{\beta}{2} - r\cos\frac{\alpha}{2}\right).$$

Then the distance of the center of gravity of the counterbalance from O is

$$\bar{x} = \frac{x_1F_1 - x_2F_2}{F_1 - F_2}$$

$$= \frac{F_2\left(r_1\cos\frac{\beta}{2} - r\cos\frac{\alpha}{2}\right)}{F_1 - F_2},$$

where $$F_1 = \frac{\alpha r^2}{2} - ar\cos\frac{\alpha}{2},$$

and $$F_2 = \frac{\beta r_1^2}{2} - ar_1\cos\frac{\beta}{2}.$$

Problem 60. If $r = 3$ ft., $r_1 = 10$ ft., and $\alpha = 120°$, find the position of the center of gravity of the counterbalance of Art. 38.

39. Graphical Method of Finding the Center of a Set of Parallel Forces in One Plane. — Let P_1, P_2, P_3, P_4 be a set of parallel forces acting at the points A_1, A_2, A_3, A_4, respectively, in one plane. Assume the forces to be acting in the direction shown in Fig. 59. Let one of the forces, as P_1, be moved to any point in its line of action, as B_1, and resolved into two components in any two directions, as $S_{5,1}$ and $S_{1,2}$. Extend the line of one of these compo-

nents, $S_{1,2}$, to cut the line of a second force, P_2, and resolve the second force into two components, one of which is equal and opposite to the component, $S_{1,2}$, of the first force on that line. Extend the line of the other component, $S_{2,3}$, of the second force, to cut the third

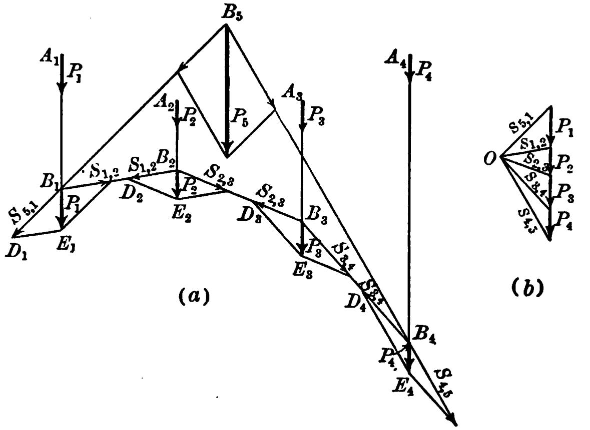


FIG. 59

force and resolve the third force into components as in the case of the second force. Proceeding in this way until the last force is reached there remain then only two unbalanced components of the original forces, one of the components, $S_{5,1}$, of the first force, and one, $S_{4,5}$, of the last force. The resultant of the original forces is therefore the resultant of these two unbalanced components. By Art. 27 the resultant of the parallel forces is parallel to the forces. Hence the resultant, P_5, acts in a line, parallel to the original forces, through the intersection, B_5, of the unbalanced components, $S_{5,1}$ and $S_{4,5}$.

The actual work of locating the line of action of the resultant may be shortened by the following consideration: the triangles of forces at the vertices B_1, B_2, B_3, B_4 may be placed with equal sides coinciding, as in Fig. 59 (*b*).

The sides representing the forces P_1, P_2, P_3, P_4 then fall in succession on a straight line, the beginning of any force falling at the end of the preceding one. The lines representing the components $S_{1,2}$, $S_{2,3}$, etc., all meet at a common point O. The rays from O are then parallel respectively to the sides of the polygon $B_1B_2B_3B_4B_5$. This enables one to draw the polygon $B_1B_2B_3B_4B_5$ without constructing the separate triangles $B_1D_1E_1$, etc. Since $S_{1,2}$ and $S_{5,1}$ had arbitrary directions given them, the point O may be chosen arbitrarily, and the polygon $B_1B_2B_3B_4B_5$ constructed by drawing its sides parallel to the corresponding rays from O. To sum up, the method is as follows:

Assume a direction in which the forces act. Lay off in succession the forces on a line parallel to their lines of action. Join their points of intersection with an arbitrary point O, Fig. 60 (*b*). Construct a polygon with vertices on the lines of action of the forces and with sides parallel to the corresponding rays from O. The intersection of the sides of this polygon, B_5 in Fig. 60 (*a*), drawn parallel to the rays from O to the beginning and end of the line of forces, is a point through which the resultant passes. This gives the line of the resultant for the assumed direction of action of the forces. The center of the forces is therefore on this line.

In the same way another line of action for another assumed direction may be determined. The intersection of these two lines is the center of the given forces (Art. 28).

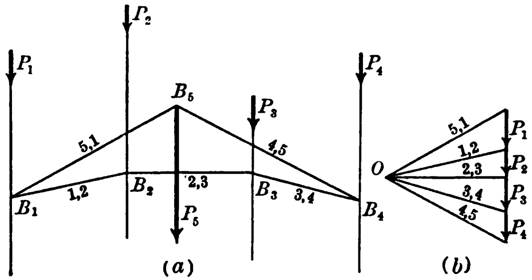


FIG. 60

In doing this work the forces may be laid off in succession in any order. It is only necessary to make the sides of the polygon $B_1B_2B_3$ ... correspond to the rays from O to the line of forces; *i.e.* the side of the polygon connecting any two forces must be parallel to the ray from O to the intersection of the same two forces.

The polygon $B_1B_2B_3$ ... is called the *equilibrium polygon.* A more general discussion will be given in the chapter on non-concurrent forces in one plane.

Problem 61. Parallel forces of 8, 12, 16, and 20 lb. act at points (0, 0), (1, 3), (3, 1), and (5, 4) respectively. Find graphically the line parallel to the y-axis in which the center of the forces lies. Make the construction twice, laying off the forces in different orders. Check by the theorem of moments.

Problem 62. Find graphically the distance from the x-axis to the center of the forces of the preceding problem.

Problem 63. Weights of 1200, 1600, 3000, and 2000 lb. act on a beam at distances of 0, 5, 8, and 12 ft. respectively from one end.

Find graphically where their resultant cuts the beam. Check by moments.

Problem 64. Locate graphically the center of gravity of the area of a quadrant of a circle.

SUGGESTION. Divide the area up into a number, say eight, of strips of equal width (Fig. 61). The weights of the strips are then approximately proportional to the ordinates at their middle points and act in the lines of these ordinates. Hence use these ordinates, or any proportional parts of them, as forces and proceed as in finding the center of a set of parallel forces.

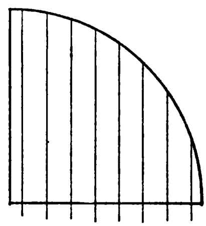


FIG. 61

Problem 65. Locate graphically the center of gravity of the arc of a quadrant of a circle.

Problem 66. Show how to find graphically, approximately, the center of gravity of any area in one plane. Apply the method to finding the center of gravity of a given area. Check by cutting the area out of cardboard and balancing.

Problem 67. Find graphically, approximately, the position of the center of gravity of a hemisphere.

SUGGESTION. The hemisphere may be divided up into a number of parts by parallel planes equally spaced. The weights of these parts will then be approximately proportional to the squares of the ordinates at the middle of the generating strips of area. Lines proportional to the squares of these ordinates may be constructed as shown in Fig. 62 (*b*). These lines may then be used as forces and the graphical solution obtained as in the preceding problems.

Problem 68. Find graphically the position of the center of gravity of a solid of revolution obtained by revolving a parabola about its axis.

Problem 69. By the method of Problem 67 find the position of the center of gravity of the frustum of a cone. Compare the result with that obtained from the answer to Problem 54 by giving particular values to R, r, and H.

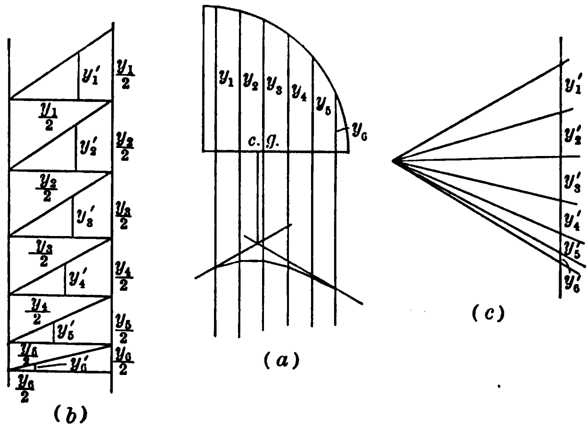


FIG. 62

40. Simpson's Rule. — When the algebraic equation of a curve is known, it is expressed as $y = f(x)$, and the area between the curve and either axis is usually determined by integration. In Fig. 63 the area $ABCD$ is expressed by the integral

$$\int_a^b y\,dx = \int_a^b f(x)\,dx,$$

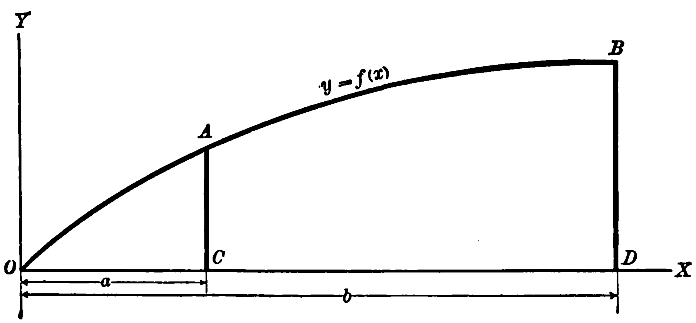


FIG. 63

when the curve represented by $y = f(x)$ is continuous between A and B.

In many engineering problems the curve is such that its equation is not known, so that approximate methods of

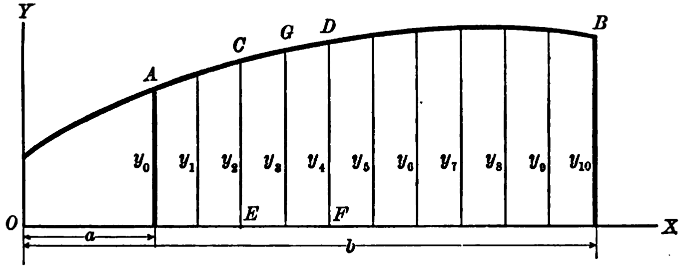


FIG. 64

obtaining the areas under the curve must be resorted to. One of these methods of approximation is known as *Simpson's Rule.* Suppose the curve in question is the curve AB (Fig. 64) and it is desired to find the area between the portion AB and the x-axis. Divide the length $b - a$ into an even number, n, of equal parts (here $n = 10$). Consider the portion $CDEF$ and imagine it magnified as shown in Fig. 65. Pass a parabolic arc through the points C, G, D, the axis of the parabola being parallel to the y-axis; then the area $CDEF$ is approximated by the area of the parabolic segment $CGDI$ plus the area of the trapezoid $CDEF$.

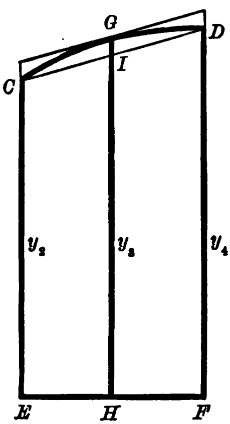


FIG. 65

$\therefore$ area $CGDEF = \frac{1}{2}(y_2 + y_4)EF + \frac{2}{3}[y_3 - \frac{1}{2}(y_2 + y_4)]EF$, since the area of the parabolic segment is $\frac{2}{3}$ the area of the circumscribing parallelogram. Since $EH = \frac{b-a}{n} = \Delta x$, this area may be written

$$\frac{1}{3}\Delta x(y_2 + 4y_3 + y_4).$$

In a similar way the next two strips to the right will have an area, $\frac{\Delta x}{3}(y_4 + 4y_5 + y_6)$, and the next two strips, an area, $\frac{\Delta x}{3}(y_6 + 4y_7 + y_8)$, and so on. Adding all these so as to get the total area under the portion of the curve AB, we get

$$\text{total area} = \frac{b-a}{3 \cdot 10}[y_0 + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8) + y_{10}],$$

or in general for n divisions,

$$\text{total area} = \frac{b-a}{3n}[y_0 + 4(y_1 + y_3 + y_5 + \cdots y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots y_{n-2}) + y_n],$$

and this is Simpson's formula for determining approximately the area under a curve. It is easy to see that the smaller Δx, the closer the approximation will be.

41. Application of Simpson's Rule. — Simpson's Rule may be made use of in determining approximately not only areas, but volumes and moments. For if a volume be divided up by planes perpendicular to the x-axis, distant Δx apart, into elements of volume Δv, then $dv = Adx$, where A is the area of the cross section at any value of x,

and the volume from $x = a$ to $x = b$ may be expressed as $\int_a^b A dx$. Hence the volume may be evaluated by Simpson's formula, the y of the formula being replaced by A.

The sum of the moments of these elements of volume with respect to the y-axis is $\int_a^b x dv$ or $\int_a^b x A dx$ and may be evaluated by Simpson's formula, replacing y by xA. On account of its use in adding moments Simpson's formula

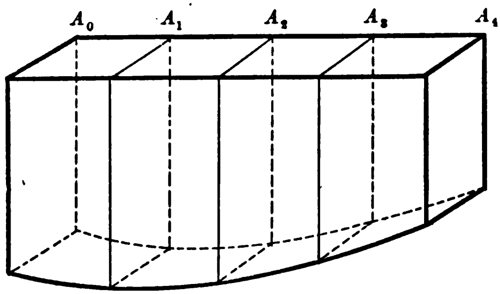


FIG. 66

may be employed in finding the center of gravity of areas or volumes bounded by lines or surfaces whose equations are not known. Suppose, for example, it is desired to know the volume and position of the center of gravity of a coal bunker of a ship as shown in Fig. 66. The bunker is 80 ft. long and the areas in square feet are as follows: $A_0 = 400$, $A_1 = 700$, $A_2 = 650$, $A_3 = 600$, $A_4 = 400$.

The distance between the successive areas is 20 ft. Applying Simpson's formula for volume,

$$\text{volume} = \frac{80}{(3)(4)}[A_0 + 4(A_1 + A_3) + 2A_2 + A_4].$$

Summing the moments, we obtain

$$\Sigma vx = \frac{80}{(3)(4)}[A_0x_0 + 4(A_1x_1 + A_3x_3) + 2A_2x_2 + A_4x_4],$$

where $x_0 = 0$, $x_1 = 20$, $x_2 = 40$, $x_3 = 60$, $x_4 = 80$. The position of the center of gravity from the fore end can now be obtained from the relation

$$\bar{x} = \frac{\Sigma vx}{v}.$$

An approximate value of $\bar{x}$ may also be obtained from the equation

$$\bar{x} = \frac{v_1x'_1 + v_2x'_2 + v_3x'_3 + v_4x'_4}{v_1 + v_2 + v_3 + v_4},$$

where $v_1 = \frac{1}{2}(A_0 + A_1)20$, $v_2 = \frac{1}{2}(A_1 + A_2)20$, etc.,

and $x'_1 = 10$, $x'_2 = 30$, etc.

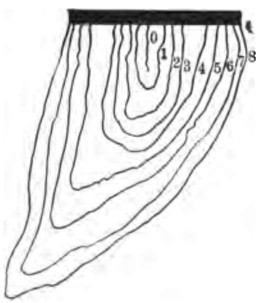


FIG. 67

Problem 70. A reservoir with five-foot contour lines is shown in Fig. 67. Find the volume of water and the distance of the center of gravity from the surface of the water, if the areas of the contour lines are as follows: $A_0 = 0$, $A_1 = 100$ sq. ft., $A_2 = 200$ sq. ft., $A_3 = 500$ sq. ft., $A_4 = 600$ sq. ft., $A_5 = 1000$ sq. ft., $A_6 = 1500$ sq. ft., $A_7 = 2000$ sq. ft., $A_8 = 2500$ sq. ft. Making substitutions in the Simpson's formula, it becomes, for the volume,

$$\text{volume} = \frac{40}{(3)(8)}[A_0 + 4(A_1 + A_3 + A_5 + A_7) + 2(A_2 + A_4 + A_6) + A_8].$$

Summing the moments by Simpson's formula, we have

$$\Sigma vx = \frac{40}{(3)(8)}[A_0x_0 + 4(A_1x_1 + A_3x_3 + A_5x_5 + A_7x_7) + 2(A_2x_2 + A_4x_4 + A_6x_6) + A_8x_8],$$

where $x_0 = 0$ ft., $x_1 = 5$ ft., $x_2 = 10$ ft., etc. Then

$$\bar{x} = \frac{\Sigma vx}{v}.$$

Both numerator and denominator are computed by Simpson's formula.

Compute $\bar{x}$ by means of the equation,

$$\bar{x} = \frac{v_1x'_1 + v_2x'_2 + v_3x'_3 + \cdots}{v_1 + v_2 + v_3 + \cdots},$$

and compare with the previous result.

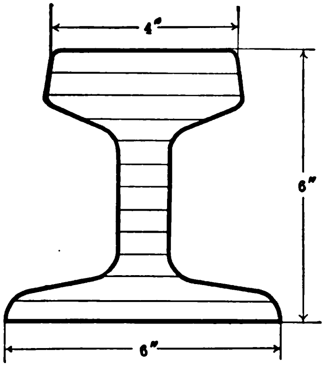


FIG. 68

Problem 71. Compute $\bar{x}$ for the parabolic area of Fig. 50, by using Simpson's Rule, and compare the result with that obtained by integration.

Problem 72. By Simpson's Rule find the area and center of gravity of the rail section shown in Fig. 68.

Beginning at the top, the horizontal distances are as follows:

$u_{12} = 4''$	$u_8 = 1.24''$	$u_4 = 1.0''$
$u_{11} = 4.08''$	$u_7 = 1.18''$	$u_3 = 1.24''$
$u_{10} = 4.24''$	$u_6 = 1.0''$	$u_2 = 2.23''$
$u_9 = 2.5''$	$u_5 = 1.0''$	$u_1 = 5.5''$
		$u_0 = 6''$

Problem 73. Find the center of gravity of the deck beam section shown in Fig. 69. Use the equation

$$\bar{x} = \frac{F_1x'_1 + F_2x'_2 + F_3x'_3 + \text{etc.}}{F_1 + F_2 + F_3 + \text{etc.}},$$

dividing the area of the section into convenient areas. Check the result thus obtained with that obtained by balancing a stiff paper model over a knife edge.

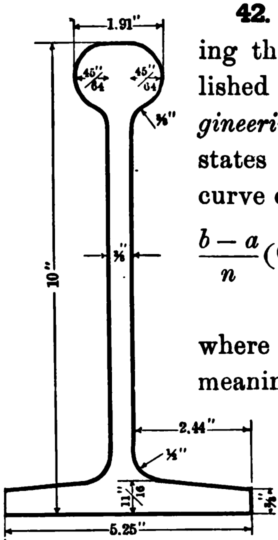


FIG. 69

42. Durand's Rule. — A method of finding the area of irregular areas was published by Professor Durand in the *Engineering News*, Jan. 18, 1894. The rule states that the total area of an irregular curve equals

$$\frac{b-a}{n}(0.4\,y_0 + 1.1\,y_1 + y_2 + y_3 + y_4 + \cdots + 1.1\,y_{n-1} + 0.4\,y_n)$$

where a, b, n, and the y's have the same meaning as in Simpson's Rule. The number of divisions may be even or odd. The student is advised to make use of this rule as well as Simpson's Rule and compare the results.

43. Theorems of Pappus and Guldinus. — Let F be a plane area and OX a line in its plane not cutting the area F (Fig. 70). Let the plane containing the area F be revolved through the angle α about the line OX as an axis. Let dF be an element of the area at a distance y from OX. Then the element of volume

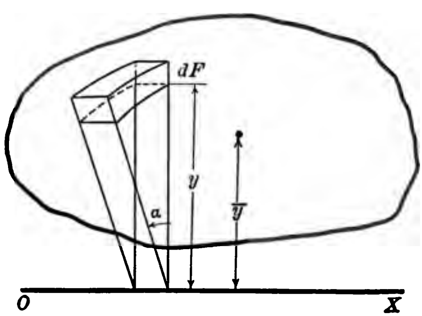


FIG. 70

generated by dF is approximately $dv = y\alpha dF$, and the

total volume generated by the area F is

$$V = \int \alpha y dF$$
$$= \alpha \int y dF.$$

But the distance of the center of gravity of the area F from OX is given by $\bar{y} = \dfrac{\int y dF}{F}$.

$$\therefore \int y dF = F\bar{y}, \text{ and}$$
$$V = F\alpha\bar{y}.$$

This may be stated as a general principle as follows:

The volume of any solid of revolution is equal to the area of the generating figure times the distance its center of gravity moves.

Problem 74. Find the volume of a sphere, radius r, by the above method, assuming it to be generated by a semicircular area revolving about a diameter.

Problem 75. Assuming the volume of the sphere known, find the center of gravity of the generating semicircular area.

Problem 76. Find the volume of a right circular cone, assuming that the generating triangle has a base r and altitude h.

Problem 77. Assuming the volume of the cone known, find the center of gravity of the generating triangle.

Problem 78. The parabolic area of Problem 40 revolves about the x-axis; find the volume of the resulting solid.

Problem 79. Find the volume of an anchor ring, if the radius of the generating figure is a and the distance of its center from the axis of revolution is r.

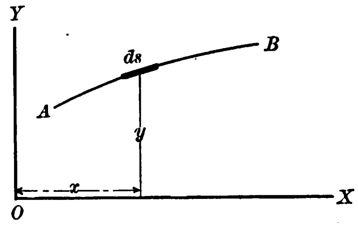


FIG. 71

Let the curve AB (Fig. 71), of length l, be the generating curve of a surface of revolution. The area of the surface generated by ds when revolved through the angle α is $dF = \alpha y ds$, and the area of the whole surface is $F = \alpha \int y ds$. The center of gravity of this curve AB is given by the expression

$$\bar{y} = \frac{\int y ds}{\int ds} = \frac{1}{l} \int y ds.$$

$$\therefore \int y ds = l\bar{y}, \text{ and}$$

$$F = l\alpha\bar{y}.$$

This may be stated as follows: *The area of any surface of revolution is equal to the length of the generating curve times the distance its center of gravity moves.*

Problem 80. Find the surface of a sphere, radius r, assuming the generating line to be a semicircular arc.

Problem 81. Find the center of gravity of a quadrant of a circular wire, radius of the circle r. Use results obtained above.

Problem 82. Find the surface of the paraboloid in Problem 78.

CHAPTER V

COUPLES

44. Definitions. — Two numerically equal forces acting in parallel lines in opposite directions form a couple. The distance between the forces is called the arm of the couple. The product of one of the forces and the arm, with the proper sign prefixed, is called the *moment of the couple.* The sign is plus if the forces tend to produce counter-clockwise rotation and negative if they tend to produce clockwise rotation. If P is one of the forces, d the distance between them, and M the moment of the couple, then $M = \pm Pd$.

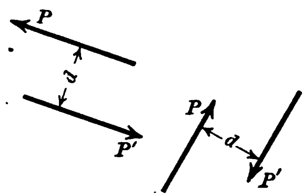


FIG. 72

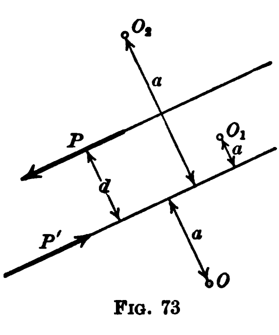


FIG. 73

The unit moment is that due to unit forces at unit distance apart. If $P = 1$ lb. and $d = 1$ ft., the moment is called 1 lb.-ft.

45. Moment of a Couple about Any Point in its Plane. — Let O be any point in the plane of the couple of Fig. 73 and let a be the distance from O to one of the forces, P', as shown.

(The forces are marked P and P' to distinguish one from the other; $P = P'$.) Then for the point O the sum of the moments of the forces P and P' with respect to the point is

$$P(a+d) - P'a, \text{ or } Pd;$$

for the position O_1 the sum of the moments is

$$P(d-a) + P'a, \text{ or } Pd;$$

for the position O_2 the sum of the moments is

$$P'a - P(a-d), \text{ or } Pd.$$

Hence, *the sum of the moments of the forces forming a couple with respect to any point in the plane of the couple is equal to the moment of the couple.*

46. Combination of Couples in the Same Plane. — Let two couples of moments M_1 and M_2 act in the same plane. The forces forming the couples may always be combined in pairs, one force from each couple, into two numerically equal, parallel forces opposite in direction, R and R' (Fig. 74). These resultant forces therefore form a couple. (If they fall in the same line, the moment of the couple is zero.)

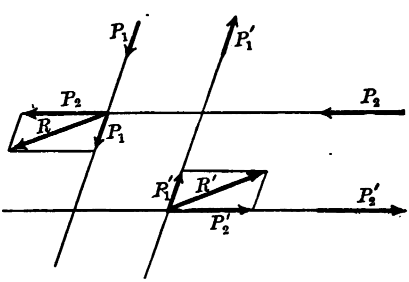


FIG. 74

Let M be the moment of this resultant couple. If any point in the plane be chosen, then with respect to this point,

$$M = \text{mom } R + \text{mom } R'.$$

But mom $R = \text{mom } P_1 + \text{mom } P_2$
and mom $R' = \text{mom } P_1' + \text{mom } P_2'$. (Art. 22.)

Therefore, adding,

$$M = (\text{mom } P_1 + \text{mom } P_1') + (\text{mom } P_2 + \text{mom } P_2')$$
$$= M_1 + M_2.$$

Therefore, *two couples in the same plane are equivalent to a single couple in their plane whose moment is equal to the algebraic sum of the moments of the two given couples.*

It follows at once that any number of couples in the same plane are equivalent to a single couple in that plane whose moment is equal to the algebraic sum of the moments of the given couples.

47. Equivalent Couples. (*a*) **In the same plane.**—Let the forces P_1 and P_1' form a couple whose moment is $M_1 = P_1 d_1$. Let any two parallel lines AB and CD distant d_2 apart cut the lines of action of P_1 and P_1' in A and C respectively. Move the forces P_1 and P_1' to A and C and at these points put in two equal and opposite forces, S and S', in the line AC, of such value that the resultant of P_1 and S falls in the line AB. The resultants, P_2 and P_2', of P_1 and S and of P_1' and S' respectively, are then numerically equal, parallel, and opposite in direction. They therefore form a couple.

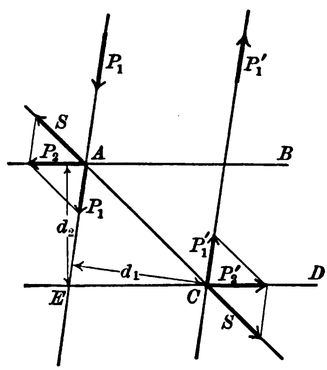


FIG. 75

By similar triangles, Fig. 75,

$$\frac{P_2}{P_1}=\frac{EC}{EA}=\frac{d_1}{d_2}.$$

Therefore $\qquad P_2d_2=P_1d_1.$

Hence, *a couple may be replaced by any other couple in its plane provided only that the moment of the second couple is equal to the moment of the first.*

(*b*) **In parallel planes.** — Let P and P' be the forces of a couple in the plane MN. From two points A and A' in their lines of action draw parallel lines to cut the plane

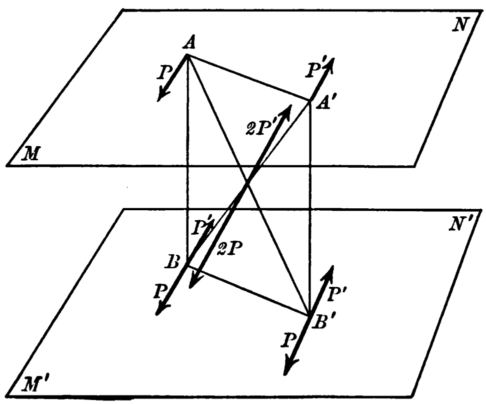


FIG. 76

$M'N'$, parallel to MN, in the points B and B'. At B and B' put in pairs of equal and opposite forces P and P' parallel to the forces P and P' in the plane MN. The two forces P and P at A and B' may be combined into a single force $2P$ acting at the middle of the line AB'. Likewise P' and P' acting at A' and B

may be replaced by the force $2P'$ acting at the middle of $A'B$. But $ABA'B'$ is a parallelogram, and the diagonals bisect each other. Therefore the forces $2P$ and $2P'$ act at the same point, and, since they are equal and opposite, annul each other. There are left then the forces P at B and P' at B', forming a couple in the plane $M'N'$, equal in moment to the original couple. Hence a couple may be moved to any plane parallel to the plane in which it acts.

To sum up (*a*) and (*b*): *A couple acting on any body may be replaced by any other couple of the same moment acting anywhere in the plane of the couple or in any parallel plane.*

48. Moment of a Couple with Respect to Any Line. Definition. — The moment of a couple with respect to any line is the sum of the moments of the forces forming that couple with respect to that line.

Let P and P' form a couple in the plane HN whose moment is $M = Pd$, and let AB be any line making the angle α with the normal to HN. Pass a plane HT perpendicular to AB to intersect the plane HN in HV. The couple may be moved in its plane until the forces are parallel to the line HV

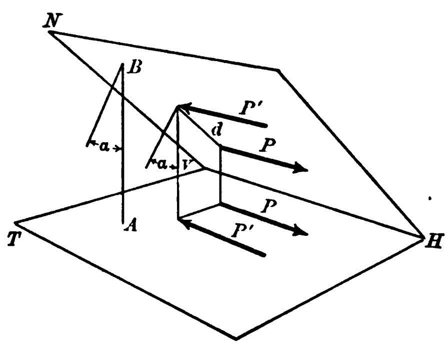


FIG. 77

(Art. 47) without changing the moment of the couple with respect to AB. For (Fig. 75) with respect to any line,

$$\text{mom } P_2 = \text{mom } P_1 + \text{mom } S$$

and

$$\text{mom } P_2' = \text{mom } P_1' + \text{mom } S'. \quad \text{(Art. 24.)}$$

Adding,

$$\text{mom } P_2 + \text{mom } P_2' = \text{mom } P_1 + \text{mom } P_1',$$

since

$$\text{mom } S + \text{mom } S' = 0.$$

The projections of the forces P and P' on the plane HT are equal to the forces themselves, and the projection of the distance, d, between them is $d \cos \alpha$. The moment of the couple with respect to AB is therefore $Pd \cos \alpha$, or $M \cos \alpha$ (Arts. 23 and 45).

49. Combinations of Couples in Intersecting Planes. — Let the couples of moments M_1 and M_2 (Fig. 78) be replaced in their planes by equivalent couples having a common arm, a, coinciding with the line of intersection of the planes (Art. 47). The forces forming the couples may then be combined into a pair of numerically equal, parallel, and opposite forces R and R' which form a couple in a plane containing the line of intersection of the two given planes.

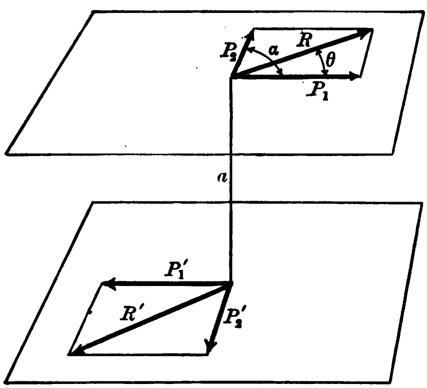


FIG. 78

If α is the angle between the planes, measured between P_1 and P_2 (Fig. 78), and M is the moment of the resultant couple, then

$$M = aR = a\sqrt{P_1^2 + P_2^2 + 2\,P_1P_2 \cos\alpha}$$
$$= \sqrt{(aP_1)^2 + (aP_2)^2 + 2\,aP_1aP_2 \cos\alpha}$$
$$= \sqrt{M_1^2 + M_2^2 + 2\,M_1M_2 \cos\alpha}.$$

The angle, θ, which the plane of the resultant couple makes with the plane of the couple of moment M_1, measured from P_1 to R, is given by

$$\tan\theta = \frac{P_2 \sin\alpha}{P_1 + P_2 \cos\alpha},$$

or

$$\tan\theta = \frac{M_2 \sin\alpha}{M_1 + M_2 \cos\alpha}.$$

Conversely, the couple of moment M may be regarded as resolved into the component couples of moments M_1 and M_2.

50. Vector Representation of Couples. — All of the properties of couples derived in the preceding articles may be represented geometrically by regarding the couple as a vector quantity as follows: A couple may be represented by a vector perpendicular to the plane of the couple, the length of the vector representing the moment of the couple. It is agreed that the vector shall point from the plane toward that side from which the rotation appears counter-clockwise.

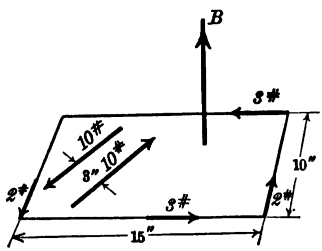


FIG. 79

Since the couple may be moved anywhere in its plane or in a parallel plane, the vector may be laid off from any point. Thus the vector AB may represent any one of the three couples shown in Fig. 79.*

The theorem of Art. 48 expressed in terms of vectors is: *The projection of the vector representing a couple upon any straight line represents the moment of the couple with respect to that line.* The theorems of Arts. 46 and 49 expressed in terms of vectors are as follows: *Couples may be combined by adding the vectors representing them; and any couple may be replaced by two or more couples whose added vectors give the vector of the original couple.*

51. Rectangular Components of a Couple. — A case of particular interest coming under the theorem just mentioned

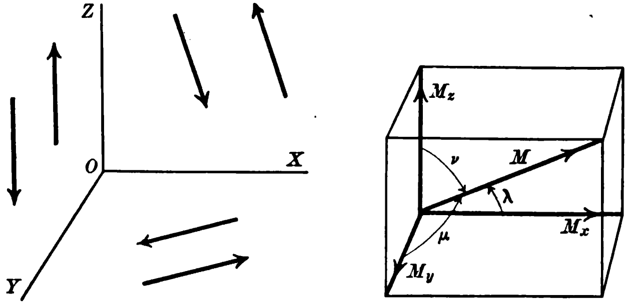


FIG. 80

is the combination of couples in three rectangular planes into a single couple and the converse problem of resolving a couple into component couples in three rectangular

* The arrow barb is placed a short distance from the end of the couple vector to distinguish it from the force vector.

planes. Let the planes be taken as coördinate planes. Call the moments of the couples in the yz-plane, the xz-plane, and the xy-plane, M_x, M_y, and M_z respectively. Representing the couples as vectors, the values M_x, M_y, and M_z are laid off along the x-, y-, and z-axes respectively (Fig. 80). The vector representing the resultant couple is then the diagonal of the rectangular parallelopiped with these vectors as edges.

If the moment of the resultant couple is M and the direction angles of the vector representing the couple are λ, μ, and ν, then

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2},$$

$$\cos\lambda = \frac{M_x}{M},\ \cos\mu = \frac{M_y}{M},\ \cos\nu = \frac{M_z}{M}.$$

Conversely, if λ, μ, ν, and M are given, the component couples in the yz-, xz-, and xy-planes have moments equal respectively to $M_x = M\cos\lambda$, $M_y = M\cos\mu$, $M_z = M\cos\nu$.

52. A Couple not Balanced by a Force.—A force and a couple cannot be in equilibrium. Let P and P' form a couple and let P_1 be a force. No matter where the force P_1 may act the couple may be moved in its own plane or in a parallel plane until one of its forces, say P, has at least one point in common with the force P_1. The forces P and P_1 then either annul each other or they have a resultant passing through their common point. This resultant and P' cannot balance each other, for two forces

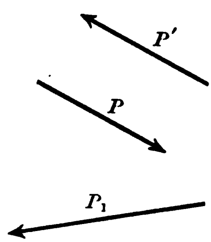


FIG. 81

can balance only when they have the same line of action and are equal and opposite. Hence the force and the couple cannot balance each other.

53. Substitution of a Force and a Couple for a Force.—Let P be any force acting at a point A and O any other point distant d from the line of P. At O put in two equal and opposite forces P and P' parallel to the force P at A. Then P at A and P' at O form a couple whose moment is Pd, and there remains the force P at O.

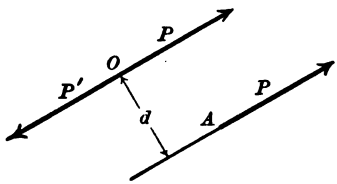


FIG. 82

Hence any force may be replaced by a force equal and parallel to the given force acting at any desired point, and a couple in the plane of the given force and the point whose moment is equal to the moment of the given force with respect to that point.

Problem 83. Replace a force of 20 lb. acting in a line 15 in. from a point O by a force acting at O and a pair of forces in parallel lines 6 in. apart in a plane parallel to the plane of the given force and the point O.

Problem 84. Parallel and opposite forces of 50 lb. each act in lines distant 20 in. apart in a plane the normal to which has direction angles $\lambda = 30°$, $\mu = 50°$, $\nu = ?$. The normal passes through the first octant, and the moment of the couple appears negative to an observer in the first octant if the plane of the couple is regarded as passing through the origin. Find the moments of this couple with respect to the coördinate axes.

Problem 85. Determine the magnitude and plane of action of the resultant couple of the couples acting on the rectangular block shown in Fig. 83.

Problem 86. Forces of 10, 15, and 20 lb. act respectively in the *xz*-, *xy*-, and *yz*-planes, parallel respectively to the *z*-, *x*-, and *y*-axes, at distances respectively of 15, 12, and 10 in. from the origin. Replace these forces by a single force acting at the origin and a couple. Find the magnitude and direction of the force vector and of the couple vector.

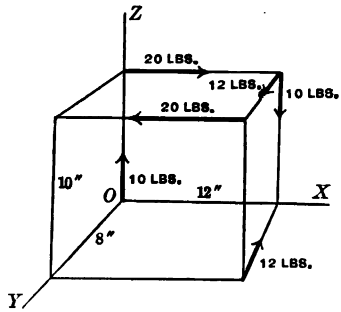


FIG. 83

Problem 87. Two forces, each equal to 10 lb., act in a vertical plane so as to form a positive couple. The distance between the forces is 2 ft. Another couple whose moment is equal to 80 lb.-in. acts in a horizontal plane and is negative. Required the resultant couple, its plane, and direction of rotation.

Problem 88. A couple whose moment is 10 lb.-ft. acts in the *xy*-plane; another couple whose moment is -30 lb.-ft. acts in the *xz*-plane and another couple whose moment is -25 lb.-ft. acts in the *yz*-plane. Required the amount, direction, and location of the couple that will hold these couples in equilibrium.

Problem 89. Show that the moment of a couple with respect to an element of a right circular cone, whose axis is perpendicular to the plane of the couple, is the same for all elements of the cone.

CHAPTER VI

NON-CONCURRENT FORCES

54. Forces in a Plane. — The most general case of forces in a plane is that one in which the forces are non-concurrent and non-parallel. We shall now consider such a

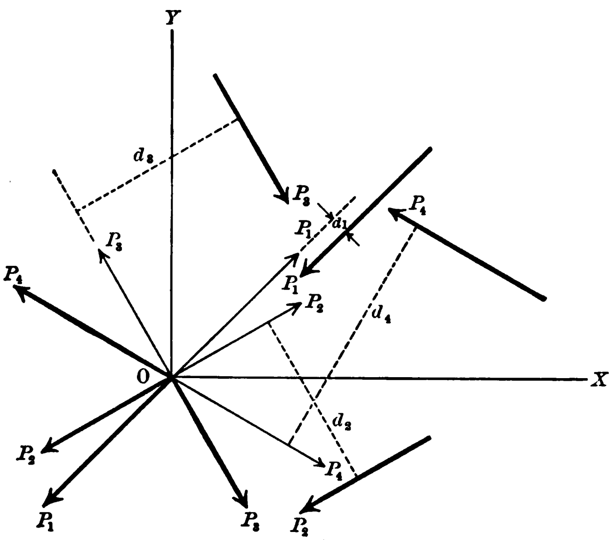


FIG. 84

case. Let the forces be P_1, P_2, P_3, P_4, etc., as shown in Fig. 84, and let them have the directions shown. Selecting arbitrarily an origin and a pair of rectangular axes in the plane of the forces, replace each force by an equal

and parallel force acting at the origin, and a couple whose moment is equal to the moment of the force with respect to the origin (Art. 53). The forces acting at the origin may then be replaced by a single force,

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}.$$

The couples, being all in one plane, are equivalent to a single couple in that plane whose moment is

$$M = \Sigma Pd. \qquad \text{(Art. 46.)}$$

Therefore the system of non-concurrent forces in a plane may be reduced to a single force, $R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$, acting at an arbitrarily selected origin, and a single couple in the plane of the forces whose moment is the sum of the moments of the forces with respect to that origin.

For equilibrium, $\boldsymbol{R = 0}$ and $\boldsymbol{\Sigma Pd = 0}$, or $\boldsymbol{\Sigma X = 0}$, $\boldsymbol{\Sigma Y = 0}$, and $\boldsymbol{\Sigma Pd = 0}$; that is, for equilibrium, the sum of the components of the forces along each of the two axes is zero and the sum of the moments with respect to any point in the plane is zero. Conversely, if $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma Pd = 0$, for a point in the plane, the resultant force R and the resultant couple both vanish and the forces are in equilibrium.

55. Other Conditions for Equilibrium of Forces in One Plane.—The forces may be combined in succession until there is obtained either a final resultant, including a zero resultant, or a couple (Arts. 13 and 25). In either case the sum of the moments of the forces with respect to

any point in the plane is equal to the moment of their resultant force or couple (Arts. 22 and 46). If the sum of the moments of the forces with respect to any selected point is zero, the forces cannot be equivalent to a couple, for the moment of a couple is the same for all points of the plane. The forces may, however, have a resultant force. But if in addition the sum of the moments of the forces about two other points of the plane not in the same straight line with the first point is zero, the resultant must be zero; for either the resultant or the arm of the resultant must be zero, and the arm cannot be zero for three different points not in the same straight line.

Hence, *if the sum of the moments of a set of forces in one plane with respect to three points not in the same straight line is zero, the forces are in equilibrium.*

It is left as an exercise for the student to prove that *if the sum of the rectangular components in one direction of a set of forces in one plane is zero and the sum of the moments of the forces with respect to each of two points in the plane not in a perpendicular to the given direction is zero, the forces are in equilibrium.*

Problem 90. Prove the statement of the last paragraph.

Problem 91. The following forces in one plane act upon a rigid body: a force of 100 lb. whose line of action makes an angle of 45° with the horizontal, and whose distance from an arbitrarily selected origin is 2 ft.; also a force of 50 lb. whose line of action makes an angle of 120° with the horizontal, and whose distance from the origin is 3 ft.; and a force of 500 lb. whose line of action makes an angle of 300° with the horizontal and whose distance from the origin is 6 ft. Find the resultant force and the resultant couple.

Problem 92. It is required to find the stress in the members *AB*, *BC*, *CD*, and *CE* of the bridge truss shown in Fig. 85.

NOTE. The member *AB* is the member between *A* and *B*, the member *CD* is the member between *C* and *D*, etc. This is a type of Warren bridge truss. All pieces (members) are pin-connected so that only two forces act on each *member*. The members are, therefore, under simple tension or compression; that is, in each member the forces act along the piece.

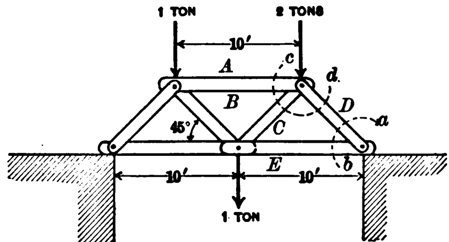


FIG. 85

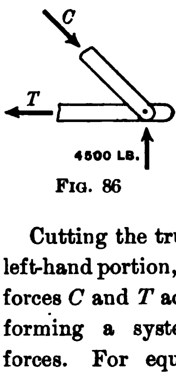


FIG. 86

SOLUTION OF PROBLEM. The reactions of the supports are found by considering all the external forces acting on the truss. Taking moments about the left support, we get the reaction at the right support, equal to 4500 lb. Summing the vertical forces or taking moments about the right-hand support, the reaction at the left-hand support is found to be 3500 lb.

Cutting the truss along *ab* and putting in the forces exerted by the left-hand portion, consider the right-hand portion (see Fig. 86). The forces *C* and *T* act along the pieces, forming a system of concurring forces. For equilibrium, $\Sigma X = 0$ and $\Sigma Y = 0$, giving two equations, sufficient to determine the unknowns *C* and *T*. The forces in the members *CD* and *CE* may now be considered known.

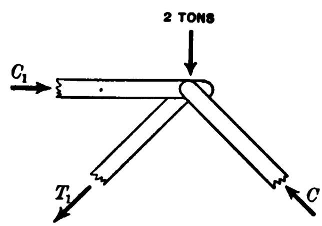


FIG. 87

Cutting the truss along the line *cd* and putting in the forces exerted by the remaining portion of the truss, we have the portion represented in Fig. 87. This gives a

system of concurring forces of which C and 2 tons are known, so that from the equations $\Sigma X = 0$ and $\Sigma Y = 0$ the remaining forces C_1 and T_1 may be found.

Problem 93. In the crane shown in Fig. 88 (*a*) find the forces acting on the pins and the tension in the tie AC. The method of cutting cannot be used in this case since the vertical and horizontal members are in flexure. Taking the horizontal member and considering all of the forces acting upon it, we have the system of non-concurring forces shown in Fig. 88 (*b*). Three unknowns are involved, P_3, P_1, and P_2, and these may be determined by three equations $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma Pd = 0$. It is to be remembered that the pin pressure at E is unknown in magnitude and direction. In all such cases it is usually more convenient to resolve this unknown pressure into its vertical and horizontal components, giving two unknown forces in known directions instead of one unknown force in an unknown direction. This will be done in all problems given here. In the present case the two forces P_1 and P_2 are the components of the unknown pin pressure.

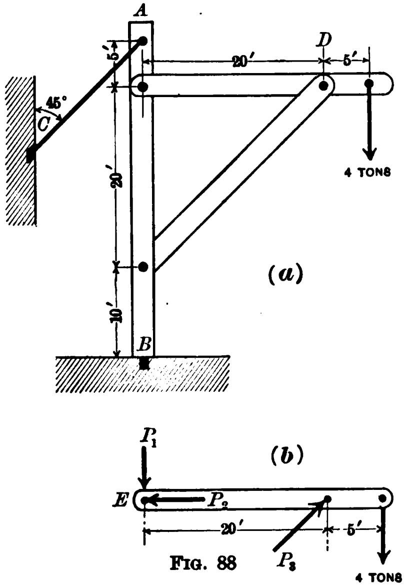


FIG. 88

The tension in the tie AC may be found by considering the forces acting on the whole crane and taking moments about B. Thus $\Sigma Pd = 0$ gives, calling the tension in the tie T,

$$T\,35 \sin 45^\circ = 8000\ (25),$$

or

$$T = \frac{8000\ (25)}{35 \sin 45^\circ}.$$

Problem 94. In the crane shown in Fig. 89 (*a*) find the forces in the ties and the compression in the boom. The method of cutting may be used here to determine the compression T and the compression in the boom, since AB is not in flexure, if we neglect its own weight. Cutting the structure about the point A and drawing the forces acting on the body, we have the system shown in Fig. 89 (*b*). The forces W may be considered as acting at the center of the pulley. The system of forces is concurring, so that $\Sigma X = 0$, and $\Sigma Y = 0$ are sufficient to determine T and C. T' may be found by considering the forces acting on the whole crane and taking moments about the lowest point. What length of AB would make the stress, T, tension?

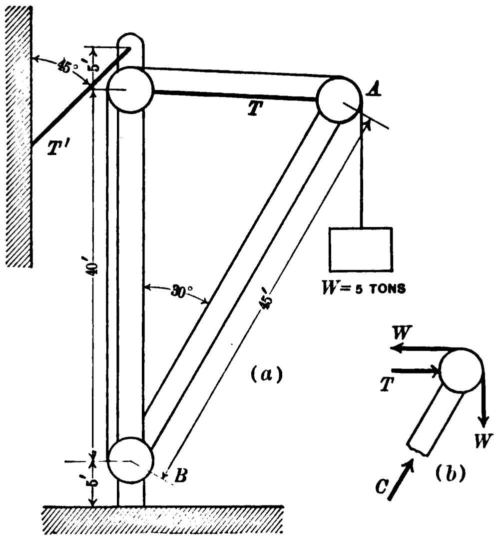


FIG. 89

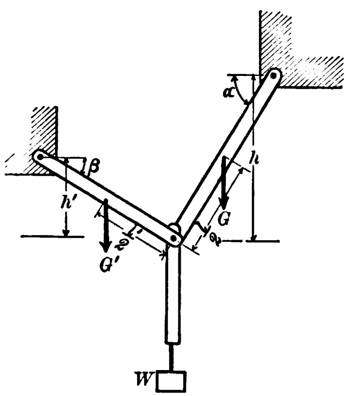


FIG. 90

NOTE. Neglecting friction, the tension W in the cord supporting the weight is transmitted undiminished throughout its length.

Problem 95. Find the horizontal and vertical components of the forces acting on the pins of the structure shown in Fig. 90.

SUGGESTION. First take the vertical member and consider all the forces acting on it.

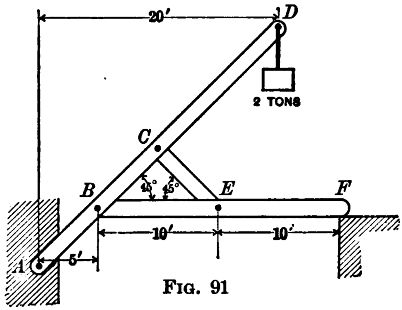


FIG. 91

Problem 96. Find the forces acting on the pins of the structure shown in Fig. 91, the weight of the members *AD*, *BF*, and *CE* being 600 lb., 400 lb., and 100 lb., respectively.

Problem 97. A traction engine is passing over a bridge, and when it is in the position shown in Fig. 92, one half of the load is carried by each truss. The weight of the engine is transmitted by the floor beams to the cross beams, and these are carried at the pin connections of the truss. Find the stress in the members *AB*, *BC*, *CE*, *CD*, and *DF*, for the position of the engine shown. The weight of the engine is 3 tons.

NOTE. The floor beams are supposed to extend only from one cross beam to another.

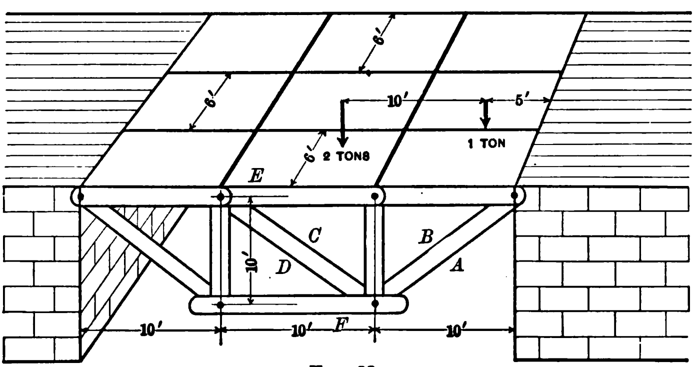


FIG. 92

Problem 98. In Problem 94, suppose the weight of the boom to be one ton; find the stresses T and T' and the pin pressures.

NOTE. The boom is now under flexure, so that the method of cutting cannot be used.

Problem 99. A dredge or steam shovel, shown in outline in Fig. 93, has a dipper with capacity of 10 tons. When the boom and dipper are in the position shown, find the forces acting on AB, CD, and EF. The projection of the point F is 6 ft. from the point E.

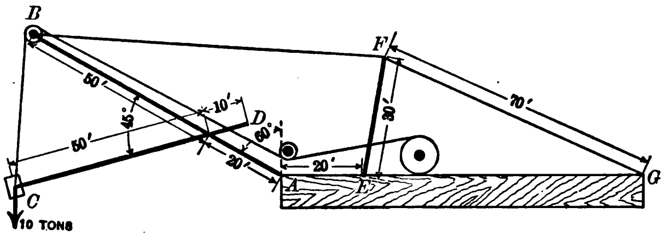


FIG. 93

SUGGESTION. Consider first all the forces acting on CD, then all the forces acting on AB.

NOTE. The member EF has been introduced as such for the sake of analysis; it replaces two legs, forming an A frame.

Problem 100. Suppose the members of the structure in Problem 99 to have weights as follows: AB, 15 tons, and CD, 3 tons, not including the 10 tons of dipper and load. Find the forces as required in the preceding problem.

Problem 101. Suppose the beam in Problem 5 to be 20 ft. long and to have a weight of 2000 lb.; find the pin reaction and the tension in the tie.

Problem 102. Assume that the compression members of the Warren bridge truss of Problem 92 have each a weight of 500 lb.; find the stresses in the members BC and CE.

56. Forces in Space. — First consider a single force P acting at A (Fig. 94). Choose any point O and rectangular axes through O as origin. Let the coördinates of

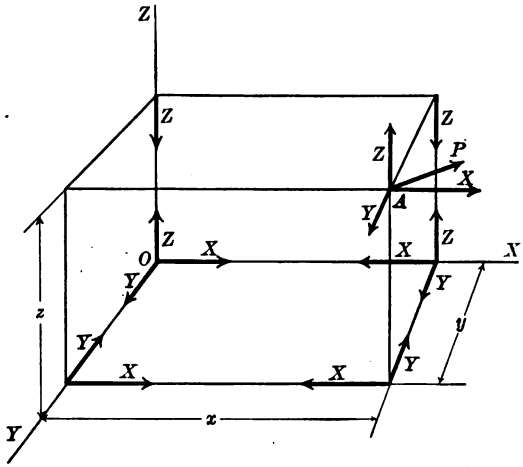


Fig. 94

A referred to these axes be (x, y, z) and let the components of P parallel to these axes be X, Y, Z. Construct a rectangular parallelopiped with O and A as opposite vertices and with x, y, and z as edges. Along these edges put in two pairs each of parallel, equal, and opposite forces, parallel to X, Y, and Z respectively, one force of each set acting at O in the direction of the corresponding component at A (Fig. 94). It is then evident, from an inspection of the figure, that the force P acting at A is equivalent to the components X, Y, Z acting at O and three couples as follows:

in the yz-plane a couple whose moment is $zY - yZ$,
in the xz-plane a couple whose moment is $xZ - zX$,
in the xy-plane a couple whose moment is $yX - xY$.

In determining the sign of the moment the observer is supposed to look toward the origin from the positive end of the axes (Fig. 95).

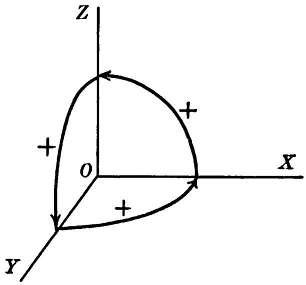


FIG. 95

From the above discussion of a single force it follows at once that any number of forces in space are equivalent to three forces,

$$\Sigma X,\ \Sigma Y,\ \Sigma Z,$$

acting along rectangular axes at an arbitrary point, and three couples in the yz-, xz-, and xy-planes whose moments are respectively

$$M_x = \Sigma(zY - yZ),\ M_y = \Sigma(xZ - zX),\ M_z = \Sigma(yX - xY).$$

The three forces acting at the origin may be combined into a single resultant,

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2},$$

whose direction cosines are

$$\cos\alpha = \frac{\Sigma X}{R},\ \cos\beta = \frac{\Sigma Y}{R},\ \cos\gamma = \frac{\Sigma Z}{R}. \qquad \text{(Art. 20.)}$$

The couples may be combined into a single couple of moment

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2},$$

whose direction cosines are

$$\cos\lambda = \frac{M_x}{M},\ \cos\mu = \frac{M_y}{M},\ \cos\nu = \frac{M_z}{M}. \qquad \text{(Art. 51.)}$$

For equilibrium both $R=0$ and $M=0$; that is,

$$\Sigma X=0,\ \Sigma Y=0,\ \Sigma Z=0,\ M_x=0,\ M_y=0,\ M_z=0.$$

Since $zY-yZ$ is the moment of the force P with respect to the x-axis, and hence M_x is the sum of the moments of all the forces with respect to the x-axis, the above conditions may be expressed in the words: *the sum of the components of the forces along each of the three arbitrarily chosen axes is zero, and the sum of the moments with respect to each of these axes is zero when the forces are in equilibrium.*

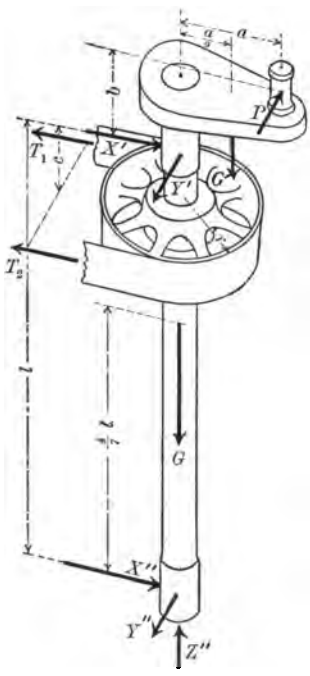


Fig. 96

Problem 103. A vertical shaft is acted upon by the belt pressures T_1 and T_2, the crank pin pressure, P, and the reactions of the supports. (See Fig. 96.) Write down the six equations for equilibrium.

Note. The y-axis has been chosen parallel to the force P, and T_1 and T_2 are parallel to the x-axis.

$$\Sigma X = X' + X'' - T_1 - T_2 = 0,$$
$$\Sigma Y = Y' + Y'' - P = 0,$$
$$\Sigma Z = Z'' - G - G' = 0;$$
$$M_x = -Pb - Y''l = 0,$$
$$M_y = X''l - T_1c - T_2c - G'\frac{a}{2} = 0,$$
$$M_z = Pa + T_1r - T_2r = 0.$$

From these six equations six unknown quantities can be found. If G, G', T_1, and T_2 are known, the reaction of the supports and P may be found.

Problem 104. In Problem 103 suppose $a=1'$, $b=8''$, $c=6''$, $l=3'$, $r=6''$, $G=90$ lb., $G'=40$ lb., $T_1=75$ lb., $T_2=200$ lb.; write

the equations for equilibrium and solve for P and the reactions of the supports.

Problem 105. Same as Problem 104 except that T_1 and T_2 make an angle of 30° (counter-clockwise when observed from top) with their directions, as shown in Fig. 96.

Problem 106. A crane shown in Fig. 97 has a boom 45 ft. long and a mast 30 ft. high. It is loaded with 20 tons, and the angle between the boom and the mast is 45°. The two stiff legs each make an angle of 30° with the mast and an angle of 90° with each other. Find the pin pressures in boom and mast, also the stress in the legs when (a) the plane of the crane bisects the angle between the legs and (b) the plane of the crane makes an angle of 30° with one of them. If the boom weighs 4000 lb., find the stress in the legs when the plane of the crane bisects the angle between them. Assume that the pulleys A and B are at the ends of the boom and mast respectively.

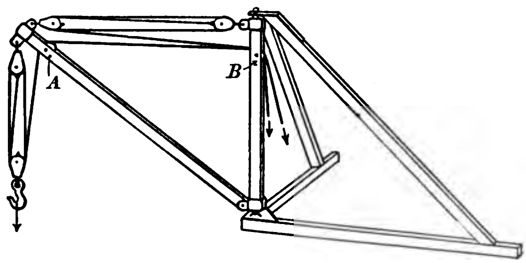


Fig. 97

Problem 107. Suppose the shaft of Problem 103 to be horizontal, find P and the reactions of the supports. Assume y horizontal and perpendicular to the shaft, and x vertical.

57. Graphical Method for Forces in One Plane. — Let P_1, P_2, P_3, P_4 (Fig. 98) be a set of forces in one plane; to find by graphical processes a force P_5 that will balance them.

Resolve P_1 into two components, $S_{1,2}$ and $S_{5,1}$, in arbitrary directions. Extend the line of $S_{1,2}$ until it meets

the line of P_2. Resolve P_2 into two components, $S_{1,2}$, equal and opposite to $S_{1,2}$ of P_1, and $S_{2,3}$. Extend the line of $S_{2,3}$ to cut the line of P_3 and proceed as before.

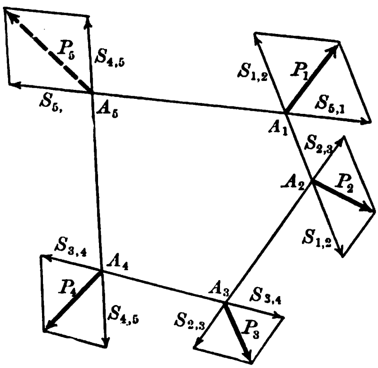


FIG. 98

When the last force, P_4, is reached, there remain unbalanced only the component, $S_{5,1}$, of the first force P_1, and $S_{4,5}$ of the last force P_4. The force P_5 that will balance these components will balance the original forces. Hence P_5 is determined in magnitude, direction, and line of action as the force that will balance these two components (Fig. 98).

The force P_5 is called the *equilibrant* of the forces $P_1 \cdots P_4$.

The polygon $A_1A_2 \cdots A_5$, with vertices on the lines of the forces, is known as an *equilibrium polygon*. If the sides of this polygon were replaced by weightless rods, or links, hinged at the vertices, the forces $P_1 \cdots P_5$ would be just balanced by the tensions (or compressions) in the rods, and the framework would retain its position. An indefinite number of equilibrium polygons may be constructed for a set of balanced forces.

A shorter method of constructing the equilibrium polygon and determining the force, P_5, to balance the given forces is the following: One triangle from each vertex

$A_1, \cdots A_5$ may be taken and their common sides put together, without changing their directions, to form a closed polygon whose sides are the vectors of the forces laid off in succession (Fig. 99). This polygon is called the *force polygon*. From the force polygon it is evident that the vector of the force P_5, the equilibrant of the given forces $P_1 \cdots P_4$, is the closing side of the polygon formed by laying off in succession the vectors of the given forces, the beginning of each vector being placed at the end of the preceding one. Having laid off the force polygon and thus determined the magnitude and direction of the equilibrant, its line of action is determined by drawing the equilibrium polygon; but this may now be done without constructing the triangles at the vertices $A_1, A_2 \cdots$. For the lines (called *rays*) from the point O in the force polygon are parallel to the corresponding sides of the equilibrium polygon, and since the directions of $S_{1,2}$ and $S_{5,1}$ were arbitrary, the point O may be any point in the plane. Join O to the vertices of the force polygon and then construct the equilibrium polygon by drawing its sides parallel to the corresponding rays of the force polygon; *i.e.* a side of the equilibrium polygon terminating on two lines of force must be parallel to the ray of the force polygon drawn to the intersection of those same two forces.

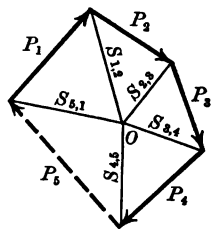


FIG. 99

To sum up: To determine graphically the equilibrant of a set of forces in one plane, construct a polygon of the

vectors of the given forces. The closing side is the vector of the equilibrant. Draw rays from any point in the plane to the vertices of the force polygon. Construct the equilibrium polygon by drawing lines parallel to the corresponding rays of the force polygon, intersecting on the lines of action of the given forces. The intersection of the two sides of the equilibrium polygon parallel to the rays to the closing side of the force polygon is a point on the line of action of the equilibrant. The equilibrant is thus completely determined.

In lettering the rays of the force polygon and the sides of the equilibrium polygon it is sufficient to mark them only with the subscripts of the forces that they connect. Figure 100 illustrates this.

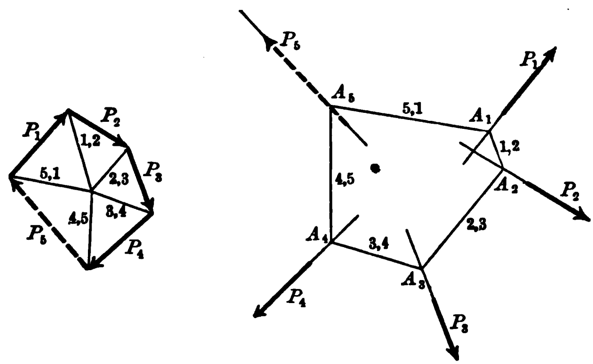


FIG. 100

When the forces are in equilibrium, as P_1, P_2, $\cdots$ P_5 in Fig. 100, both the force polygon and the equilibrium polygon are closed. It is possible to have the force polygon close without having the equilibrium polygon close; as,

for example, $P_1, \cdots P_4$ and a force P_5 parallel and equal to the closing side of the force polygon but not passing

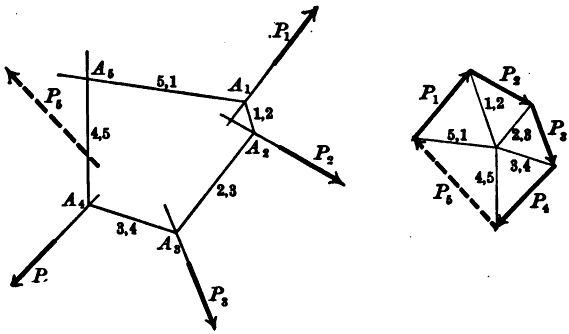


FIG. 101

through the remaining vertex of the equilibrium polygon (Fig. 101). The forces in this case form a couple, for the forces $P_1, \cdots P_4$ are equivalent to a force equal and opposite to P_5 passing through the vertex A_5 of the equilibrium polygon.

Problem 108. Determine graphically the equilibrant of the four forces of Fig. 102. Where does it cut the line AB?

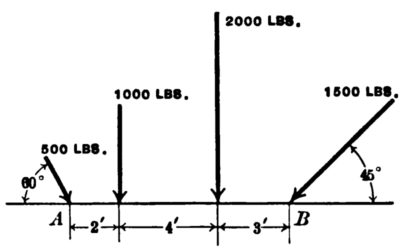


FIG. 102

Problem 109. Find graphically the reactions due to the loads in Fig. 103.

SUGGESTION. The force polygon here becomes a straight line. Call the reactions P_4 and P_5, and agree that P_4 shall follow P_3 in the force polygon. P_5 must then close the polygon. The missing ray to the intersection of P_4 and P_5 is found by drawing from O a ray par-

allel to the closing side (dotted) of the equilibrium polygon. Hence P_4 and P_5 are determined. Check by moments.

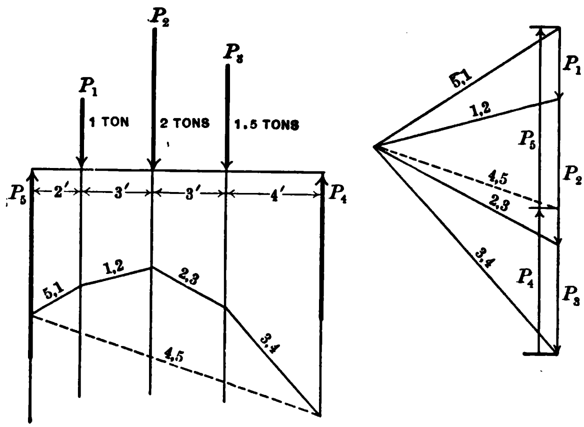


FIG. 103

When a truss is acted upon by wind loads, the directions of the supporting forces depend upon the manner in which the truss is attached to the supports. Some trusses are pinned at one end and rest on rollers at the other. The reaction at the roller end may then be assumed to be vertical, the pins taking all of the horizontal load. If the truss is fixed at both ends, the reactions may be assumed to be in parallel lines, or else that the supports take equal parts of the horizontal load.

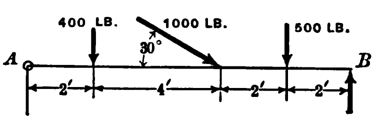


FIG. 104

Problem 110. The beam AB in Fig. 104 carries loads as shown. Find graphically the reactions at A and B, given that the reaction at B is vertical.

SUGGESTION. Since the point A is the only known point on the reaction through A, begin the equilibrium polygon at A as a vertex.

Problem 111. Find graphically the reactions of the truss of Fig. 105, assuming that the reactions are in parallel lines.

Problem 112. Find graphically the reactions of the truss of Fig. 105, assuming that the left-hand reaction is vertical.

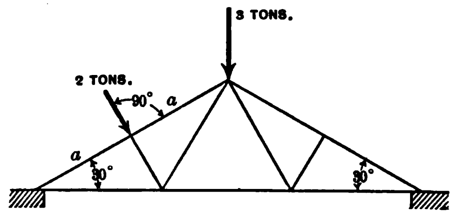


FIG. 105

58. Stresses in Frames. — Stresses in roof and bridge trusses are usually computed on the supposition that the members are two force pieces; *i.e.* are pin-connected at the ends and have loads applied only at the joints. The stresses in the truss of Problem 92 were computed on that supposition. The graphical method of determining the stresses is often more easily and quickly applied than the analytical method, except for very simple cases. As an illustration of the graphical method, the stresses in the truss of Fig. 106 are determined.

It is convenient here to use *Bow's notation.* Represent a force acting on the truss, or a member of the truss, by a pair of letters, one on each side of the line of the force or member. Thus the left-hand load on the truss is read AB, the right-hand reaction is CD, the upper horizontal member of the truss is BF, etc. Determine the reactions, either graphically or by moments. They are $CD = \frac{17}{4}$, $DA = \frac{11}{4}$. The loads and reactions form a system in equilibrium. These forces are laid off on a vertical line,

known as the load line, in succession in the order in which they are met in going around the outside of the truss, as *AB*, *BC*, *CD*, *DA*. The force *AB* is represented on the load line by the same letters in small type, *ab*, etc. At any joint of the truss three or more forces act in the

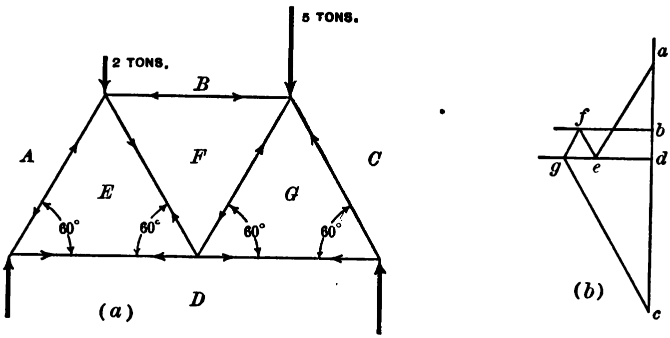


FIG. 106

directions of the forces and members that meet at that joint. These forces are in equilibrium and must therefore form a closed polygon when laid off in succession (Art. 16). In constructing the polygons the forces already laid off on the load line are made use of. Also a polygon of forces, all of which are known in direction, cannot be constructed if more than two forces are unknown in magnitude. Hence the force polygon is first drawn for a joint where only two forces are unknown. The determination of the forces acting at this point gives additional known forces at other points, for the force in any member acts equally and in opposite directions on the pins at its ends.

Choosing the joint of the truss at the left support, the forces *DA*, *AE*, and *ED* must form a closed triangle. Hence using the force *da* on the load line, complete the force triangle *dae* by drawing through *d* and *a* lines parallel respectively to *DE* and *AE* to meet in *e*. The lines *ae* and *ed* represent the forces in *AE* and *ED*. The directions of the forces acting at the joint are those given by passing around the triangle *dae* in the order *d-a-e*, since the force *da* acts from *d* towards *a*. Next go to the joint *ABFE* and construct the force polygon for that joint, using the known forces *ea* and *ab*, drawing lines through *e* and *b* parallel respectively to *EF* and *BF* to intersect in *f*. The polygon *abfea* is then the force polygon for the joint *ABFE*. The directions of the forces acting at the joint are given by passing around the polygon in the order *a-b-f-e-a*, since the force *ab* acts from *a* toward *b*. The direction in which the forces act at the joint are indicated by putting arrows on the members near the joint. Thus the member *AE* pushes against the pins at its ends, while *ED* pulls on the pins at its ends. *AE* is in compression, *ED* in tension.

The whole stress diagram (Fig. 106 (*b*)) is thus constructed upon the load line. The stress in any member is represented by the line in the force diagram terminating in the same letters as those of the member, as *fg* represents the force in *FG*.

A check on the correctness of the work is that the last line of the force diagram must be parallel to the corresponding member of the truss. More accurate results are obtained by drawing the figure to large scales.

Problem 113. In the truss of Fig. 107 the apex is distant $\frac{1}{3}$ the length of the span above the lower chord. The panels carry equal loads, half the load of each panel being transferred to the truss at the

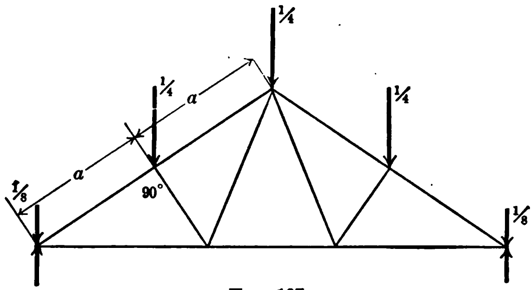


FIG. 107

pin at the end of that panel. Calling the total load unity, find graphically the stresses in the members of the truss. Write the values of the stresses on the figure of the truss and indicate whether tension or compression.

Problem 114. A wind load acts on the truss of Fig. 107, as shown in Fig. 108. Assuming the reactions to be in parallel lines, find the reactions and stresses in the members graphically, and write down the stresses as in the preceding problem.

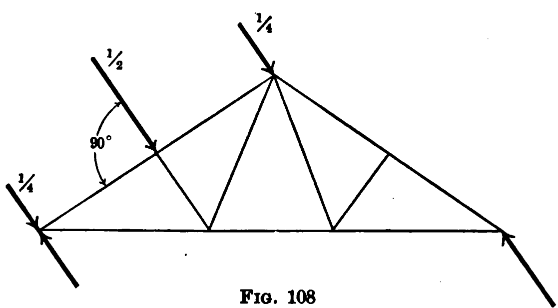


FIG. 108

SUGGESTION. In finding the reactions the work will be simplified by combining the loads into one load.

Problem 115. Assuming the total wind load to be equal to $\frac{1}{2}$ of the total vertical load in the preceding problems find the stresses when both loads act simultaneously.

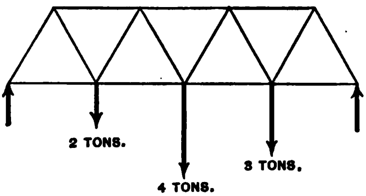


FIG. 109

Problem 116. Find the stresses in the truss of Fig. 109, all members of the truss being of the same length.

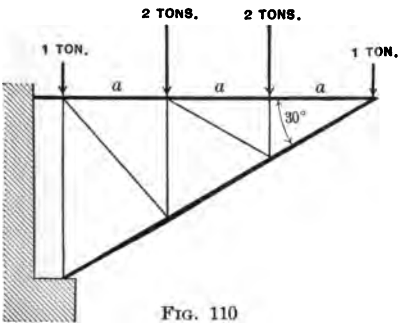


FIG. 110

Problem 117. Find the stresses in the members of the cantilever truss of Fig. 110.

Problem 118. Find the stresses in the truss of Fig. 111.

59. Method of Substitution. — It sometimes happens that in constructing the force diagram a point is reached where it

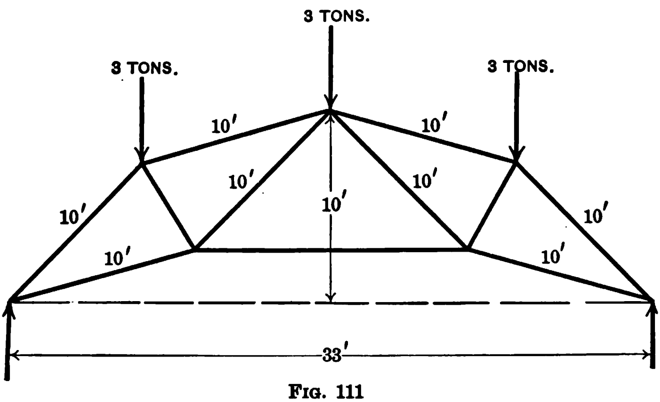


FIG. 111

is not possible to proceed to another joint at which there are only two unknown forces acting. The temporary removal of two members and the substitution of another member to hold the truss from collapsing will sometimes enable one to continue the construction of the force diagram. This is illustrated in the case of the Fink truss shown in Fig. 112.

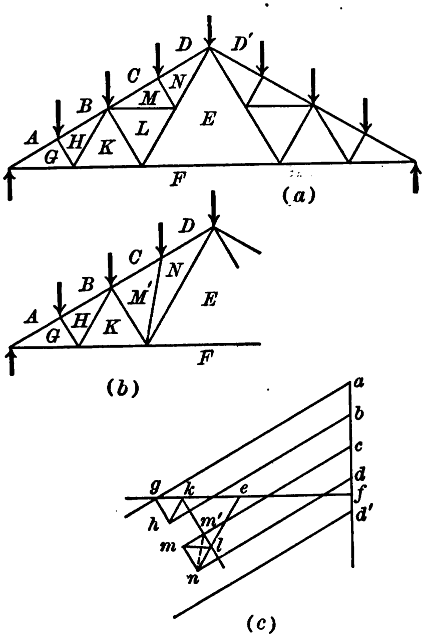


FIG. 112

Starting at the left support, the force polygons *fag*, *abhg*, and *fghk* can be constructed. The polygons for the remaining joints in the left half of the truss then have three sides missing and cannot be completed in the usual way. If the members *ML* and *MN* be removed, and a member *M′N* (Fig. 112(*b*)) be substituted, the truss would stand. Moreover, the stresses in *DN*, *NE*, and *EF* would not be changed; for if sections were made across these members in the two cases, the parts of the truss to the left of the cuts would be acted upon by the same forces, and hence would require the same forces in *DN*, *NE*, and

EF to balance them. (The same is true for sections across *BH*, *HK*, and *KF*.) With the changed form of the truss the force polygons *khbcm′* and *m′cdn* can be constructed. Having located *n*, we may return to the original truss and construct the polygon *nmcd*, then *bcmlkh*, etc.

Problem 119. The panels of a Fink truss are equal and carry equal loads. The upper chord is inclined 30° to the horizontal, and the lower chord is horizontal. Calling the total load unity, construct the stress diagram and write the stresses in the members on the figure of the truss, indicating tension and compression.

CHAPTER VII

MOMENT OF INERTIA

60. Definition of Moment of Inertia. — The study of many problems in mechanics brings to our attention the value of the integral of the form $\int y^2 dF$, where dF represents an infinitesimal area and y is the distance of that element from an axis of reference. The value of this integral taken over a given area is called the moment of inertia of that area with respect to the given axis of reference. In like manner $\int r^2 dV$ and $\int r^2 dM$ taken to include all elements of a given volume or a given mass are called respectively the moment of inertia of the volume, and mass, with respect to the given axis. In these integrals dV and dM represent infinitesimals of volume and mass respectively, and r the distance of the infinitesimal element from the axis of reference. We shall designate moment of inertia by the letter I. Thus we write :

$$I = \int y^2 dF,$$

$$I = \int r^2 dM,$$

$$I = \int r^2 dV,$$

for area, mass, and volume, respectively.

Many problems that confront the engineer involve in their solution the consideration of the moment of inertia. This is the case when the energy of a rotating flywheel, for example, is being determined. The energy of a rotating body (Art. 137) is expressed as follows:

$$\text{Kinetic energy} = \frac{I\omega^2}{2},$$

where I is the moment of inertia with respect to the axis of rotation and ω is the angular velocity. (See Art. 118.) It is seen that the energy of rotating bodies, having the same angular velocity, or the same speed, is directly proportional to their moments of inertia. The quantity, therefore, plays a very important part in the consideration of rotating bodies.

Also the strength of a beam or column depends upon the moment of inertia of the area of the cross section of the beam or column with respect to a line of the section through the center of gravity of the section.

In a later chapter a reason will be seen for the name "moment of inertia." For the present it may be regarded as a name arbitrarily applied to a quantity frequently used in the applications of mechanics. (See Art. 137.)

61. Radius of Gyration. —The moment of inertia of an area involves area times the square of a distance. We may write $I = \int y^2 dF = Fk^2$, where F is the area and k is a distance, at which, if the area were all situated, the moment of inertia would be unchanged. This distance k is called the *radius of gyration*. In a similar way

for a mass we write : $I=\int y^2 dM = Mk^2$, and for volume $I=\int y^2 dV = Vk^2$.

62. Units of Moment of Inertia. — The moment of inertia of an area with respect to any axis may be expressed as Fk^2. The area involves square inches, and k^2 is a distance squared. The product is expressed as inches to the fourth power. The moment of inertia of a volume Vk^2 requires inches to the fifth power. The moment of inertia of a mass requires in addition to Vk^2 the factor $\frac{\gamma}{g}$, so that pounds and feet per second per second are involved. This is somewhat more complicated since it involves units of weight, distance, and time. The presence of g ($= 32.2$) in the expression requires that all distances be in feet. It is customary to express the moment of inertia of a mass without designating the units used, it being understood that feet, pounds, and seconds were used.

63. Representation of Moment of Inertia. — From the definition of moment of inertia it is evident that an area has a different moment of inertia for every line in its plane. We shall designate the moment of inertia with respect to a line through the center of gravity by I_g with a subscript to indicate the particular gravity line intended. For example, I_{gx} indicates the moment of inertia with respect to a gravity axis parallel to x, and I_{gy} indicates the moment of inertia with respect to a gravity axis parallel to y. The moment of inertia with respect to a line other than a gravity line will be designated by I, the proper

subscript indicating the particular line. Similar subscripts will be used to designate the corresponding radii of gyration. It should be noted that moment of inertia is not a quantity involving direction. It has to do only with *magnitude* and is essentially *positive.*

64. Moment of Inertia. Parallel Axes.

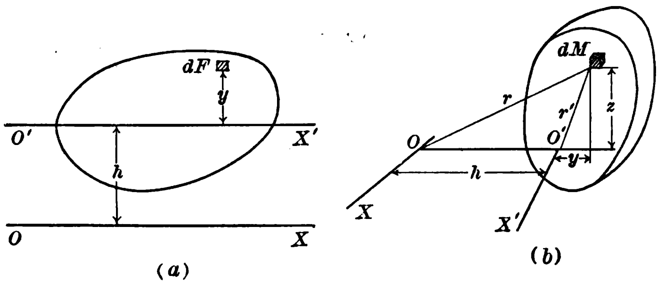


FIG. 113

(*a*) *For Areas.* In Fig. 113 (*a*), let OX and $O'X'$ be two parallel axes in the plane of the area F. If y is the ordinate of an element of area dF referred to $O'X'$ and h the distance between the axes, then

$$\begin{aligned} I_x &= \int (y+h)^2 dF = \int (y^2 + 2\,hy + h^2) dF \\ &= \int y^2 dF + 2\,h \int y dF + h^2 \int dF \\ &= I'_x + 2\,h\bar{y}F + h^2 F, \end{aligned}$$

where $\bar{y}$ is the ordinate of the center of gravity of the area F referred to the axis $O'X'$ (Art. 36).

If $O'X'$ passes through the center of gravity of F, then $\bar{y}$ is zero and the equation may be written

$$I_x = I_g + h^2 F,$$

where I_g is the moment of inertia of F with respect to a gravity axis parallel to OX. This relation may be expressed in words as,

The moment of inertia of an area with respect to any line in its plane is equal to its moment of inertia with respect to a parallel gravity axis plus the area times the square of the distance between the two axes.

(*b*) *For Solids.* In Fig. 113 (*b*), OX and $O'X'$ are parallel axes distant h apart. dM is the mass of an element of the body distant r from OX and r' from $O'X'$. If the coördinates of dM are (y, z), as shown in the figure, then

$$\begin{aligned} r^2 &= (h+y)^2 + z^2 \\ &= h^2 + 2\,hy + r'^2. \end{aligned}$$

Therefore $$\begin{aligned} I_x &= \int r^2 dM = \int (h^2 + 2\,hy + r'^2) dM \\ &= h^2 \int dM + 2\,h \int y dM + \int r'^2 dM \\ &= h^2 M + 2\,h\bar{y}M + I'_x. \end{aligned}$$

If $O'X'$ passes through the center of gravity of the solid, $\bar{y} = 0$ and the equation reduces to

$$I_x = I_g + h^2 M.$$

From the above formulæ it is evident that the moment of inertia of an area or a solid about a gravity axis is less than that for any parallel axis.

65. Moment of Inertia; Inclined Axis. — It is often desirable, when I_x and I_y are known, to find the moment of inertia with respect to an axis w making an angle α with x. (See

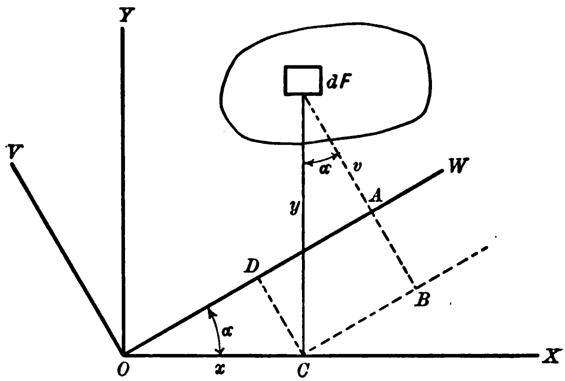


FIG. 114

Fig. 114.) Here, $I_w = \int v^2 dF$ and $I_v = \int w^2 dF$. In terms of x, y, and α,

$$I_w = \int (y \cos \alpha - x \sin \alpha)^2 dF$$
$$= \int y^2 \cos^2 \alpha \, dF - 2\int xy \cos \alpha \sin \alpha \, dF + \int x^2 \sin^2 \alpha \, dF$$
$$= \cos^2 \alpha \int y^2 dF - 2 \sin \alpha \cos \alpha \int xy dF + \sin^2 \alpha \int x^2 dF$$
$$= I_x \cos^2 \alpha - \sin 2\,\alpha \int xy dF + I_y \sin^2 \alpha.$$

In a similar way

$$I_v = I_x \sin^2 \alpha + 2 \sin \alpha \cos \alpha \int xy dF + I_y \cos^2 \alpha,$$

where OV is perpendicular to OW.

These are the required formulæ for obtaining the moment of inertia with respect to inclined axes. It follows that

$$\boldsymbol{I_w + I_v = I_x + I_y}.$$

That is, the sum of the moments of inertia of an area with respect to two rectangular axes in its plane is the same as the sum of the moments of inertia with respect to any other two rectangular axes in the same plane and passing through the same point. This states that the sum of the moments of inertia for any two rectangular axes through a point is constant. It will be seen in Art. 68 that this constant is the *polar* moment of inertia.

66. Product of Inertia. — The integral $\int xydF$ is called a *product of inertia*, for want of a better name. In case the area has an axis of symmetry, either the x- or y-axis may be taken along such an axis. The product of inertia then becomes zero, since if x is the axis of symmetry, for every $+y$ there is a corresponding $-y$. A similar reasoning shows that the product of inertia is zero when y is the axis of symmetry. In such cases

$$I_w = I_x \cos^2 \alpha + I_y \sin^2 \alpha$$

and

$$I_v = I_x \sin^2 \alpha + I_y \cos^2 \alpha.$$

When $\int xydF$ is not equal to zero, it is necessary to select the proper limits of integration and sum the integral over the area in question. This is illustrated in Art. 76.

67. Axes of Greatest and Least Moment of Inertia. — It is often important to know for what axis through the center of gravity the moment of inertia is least or greatest; that is, what value of α makes I_w or I_v a maximum or a minimum. For any area I_x, I_y, and $\int xydF$ are constant after

the x- and y-axes have been selected. Using the method of the calculus for finding maxima and minima, we have, putting

$$\int xydF = i,\ \frac{dI_w}{d\alpha} = (I_y - I_x)\sin 2\alpha - 2i\cos 2\alpha.$$

Equating the right-hand side to zero, the value of α that gives either a maximum or minimum is seen to be given by the equation

$$\tan 2\alpha = \frac{2i}{I_y - I_x},$$

or, what is the same thing,

$$\sin 2\alpha = \frac{2i}{\pm\sqrt{4i^2 + (I_y - I_x)^2}} \text{ and } \cos 2\alpha = \frac{I_y - I_x}{\pm\sqrt{4i^2 + (I_y - I_x)^2}}.$$

It is seen upon substituting these values of $\sin 2\alpha$ and $\cos 2\alpha$ in

$$\frac{d^2 I_w}{d\alpha^2} = 2(I_y - I_x)\cos 2\alpha + 4i\sin 2\alpha$$

that the positive sign before the radical indicates a minimum and the negative sign a maximum value for I_w.

The equation for $\tan 2\alpha$ is satisfied by two values of 2α differing by 180°. These values of 2α correspond to the two signs in the values for $\sin 2\alpha$ and $\cos 2\alpha$. The values of α corresponding to the two signs therefore differ by 90°. Hence, through any point of the plane, there are two axes at right angles to each other about one of which the moment of inertia is a maximum and about the other a minimum. These axes are known as the *principal axes* of the area through that point. The axes through the center of

gravity of the area for which the moments of inertia was greatest and least are known as the *Principal Axes of the Area.* This subject will be further discussed in Art. 76.

It is seen from the above that when $\int xydF = 0$, the values of α which give maximum or minimum values for I_w and I_v are 0° and 90°. This means that the x- and y-axes, themselves, are the principal axes. Conversely, if the x- and y-axes are principal axes, then $\int xydF = 0$. If either the x- or y-axis is an axis of symmetry, $\int xydF = 0$ and hence the x- and y-axes are principal axes.

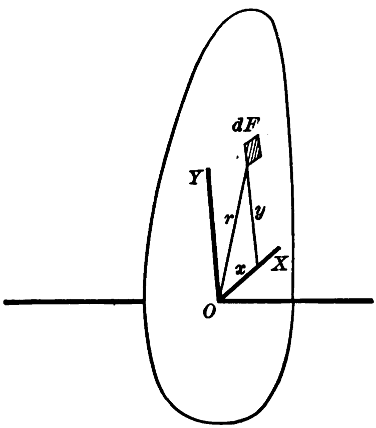


FIG. 115

Problem 120. Derive the formula for I_v written in Art. 65.

Problem 121. Prove that when I_w is a maximum, I_v is a minimum; and that when I_w is a minimum, I_v is a maximum, by using

$$I_w + I_v = \text{constant}.$$

68. Polar Moment of Inertia.—The moment of inertia of an area with respect to a line perpendicular to its plane is called the *polar moment of inertia* of the area.

Consider the area represented in Fig. 115 and let the axis be perpendicular to the area at any point O. Let dF represent an infinitesimal area and let r be its distance from the axis. Representing the polar moment of inertia

by I_p, we have

$$I_p = \int r^2 dF;$$

but $$r^2 = x^2 + y^2,$$

so that $$I_p = \int x^2 dF + \int y^2 dF,$$

or $$\boldsymbol{I_p = I_y + I_x}.$$

That is, *the polar moment of inertia of an area is equal to the sum of the moments of inertia of any two rectangular axes through the same point.*

It has already been shown that $I_x + I_y =$ constant (see Art. 65) for any point of an area.

69. Moment of Inertia of Rectangle.—Let it be required to find the moment of inertia of the rectangle shown in Fig. 116 (*a*), with respect to the axis x. We may write

$$I_x = \int y^2 dF.$$

Since $dF = bdy$, this becomes

$$I_x = b\int_0^h y^2 dy = \frac{bh^3}{3}.$$

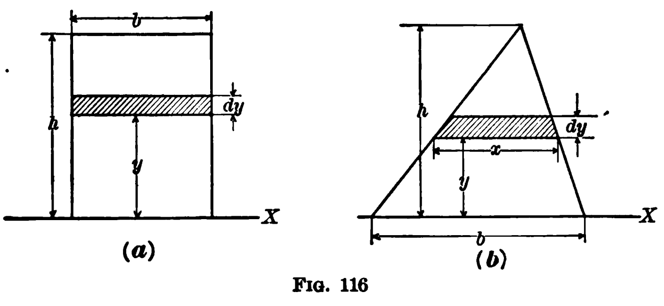


FIG. 116

To find the moment of inertia with respect to a gravity axis parallel to x we may make use of the formula $I_{gx} = I_x - Fd^2$, from which we have

$$I_{gx} = \frac{1}{12} bh^3 \text{ and } k^2_{gx} = \frac{h^2}{12}.$$

The same result may of course be obtained by taking the axes through the center and integrating between the limits $-\frac{h}{2}$ and $\frac{h}{2}$. From comparison we may write the moment of inertia with respect to a gravity line perpendicular to x,

$$I_{gy} = \frac{1}{12} hb^3 \text{ and } k^2_{gy} = \frac{b^2}{12},$$

and the polar moment of inertia for the center of gravity

$$I_p = \frac{bh}{12}(h^2 + b^2).$$

$$k^2_p = \frac{h^2 + b^2}{12}.$$

70. Moment of Inertia of a Triangle. — It is required to find the moment of inertia of the triangle shown in Fig. 116 (*b*) with respect to the axis x, coinciding with the base of the triangle. We have

$$I_x = \int y^2 dF, \text{ where } dF = x dy,$$

$$I_x = \int_0^h y^2 x dy.$$

But $x = \frac{b}{h}(h - y)$, from similar triangles, giving

$$I_x = \frac{b}{h}\int_0^h y^2(h - y)dy = \frac{1}{12} bh^3 \text{ and } k^2_x = \frac{h^2}{6}.$$

The moment of inertia with respect to the horizontal gravity axis may now be determined.

$$I_{gx} = I_x - Fd^2 = \frac{bh^3}{36}, \text{ and } k_{gx}^2 = \frac{h^2}{18}.$$

The same results may of course be obtained by direct integration.

Problem 122. Find the moment of inertia of the area of a triangle with respect to an axis through the vertex parallel to the base.

Problem 123. Find the polar moment of inertia of the area of a *right* triangle for the center of gravity.

71. Moment of Inertia of a Circular Area. — The moment of inertia of a circular area with respect to a horizontal gravity axis x, as shown in Fig. 117, may be found as fol-

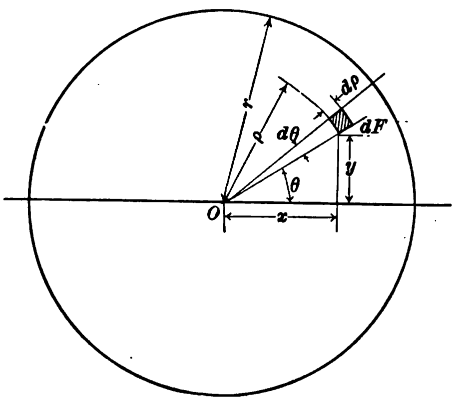


FIG. 117

lows: $I_{gx} = \int y^2 dF$. Here it is simpler to use polar coördinates. Changing to polar coördinates, remembering that

$y = \rho \sin \theta$, and $dF = d\rho(\rho d\theta)$, the integral becomes

$$I_{gx} = \int_0^{2\pi}\int_0^r \rho^2 \sin^2 \theta \rho d\theta d\rho$$
$$= \frac{r^4}{4}\int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2\,\theta)d\theta$$
$$= \frac{r^4}{4}\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\,\theta\right]_0^{2\pi} = \frac{\pi r^4}{4}.$$

The corresponding radius of gyration is $k_{gx} = \frac{r}{2}$. On account of the symmetry of the figure this is the moment of inertia for any line in the plane through the center of gravity. It follows that

$$\begin{cases} I_{gy} = \dfrac{\pi r^4}{4} \\ k_{gy}^2 = \dfrac{r^2}{4} \end{cases} \text{and that } \begin{cases} I_p = \dfrac{\pi r^4}{2}. \\ k_p^2 = \dfrac{r^2}{2}. \end{cases}$$

72. Moment of Inertia of Elliptical Area. — Let it be required to find I_{gx} and I_{gy} of the elliptical area shown in Fig. 118. The equation of the bounding curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and $$I_{gy} = \int x^2 dF = \int x^2\, 2\,y dx.$$

From the equation of the bounding curve

$$y = \frac{b}{a}\sqrt{a^2 - x^2},$$

so that

$$I_{gy} = \frac{4\,b}{a}\int_0^{+a} x^2 \sqrt{a^2 - x^2}\, dx$$

$$= \frac{4\,b}{a}\left[\frac{x}{8}(2\,x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8}\sin^{-1}\frac{x}{a}\right]_0^{+a}$$

$$= \frac{ba^3\pi}{4}, \text{ and therefore, } k_{gy} = \frac{a}{2}.$$

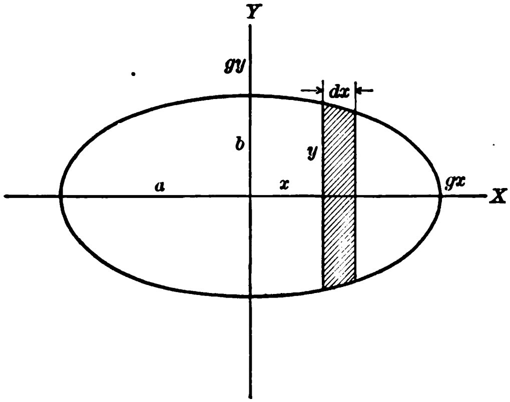


FIG. 118

In a similar way

$$I_{gx} = \int y^2 dF = \int y^2\, 2\, xdy$$

$$= \frac{4\,a}{b}\int_0^{+b} y^2\sqrt{b^2 - y^2}\, dy = \frac{ab^3\pi}{4}, \text{ and therefore } k_{gx} = \frac{b}{2}.$$

Since $I_p = I_{gx} + I_{gy}$, the polar moment of inertia with respect to the center is

$$\frac{ab\pi}{4}(a^2 + b^2), \text{ and } k_p \doteq \tfrac{1}{2}\sqrt{a^2 + b^2}.$$

It is seen that when $a = b = r$, the equations obtained for the elliptical area are the same as those obtained for the circular area, just as they should be.

73. Moment of Inertia of Angle Section. — When an area may be divided up into a number of triangles, or rectangles, or other simple divisions, the moment of inertia of the whole area with respect to any axis is often most easily found by taking the sum of the moments of inertia of the individual parts. This method is often made use of in determining the moment of inertia of such areas as the section of the angle iron, shown in Fig. 119.

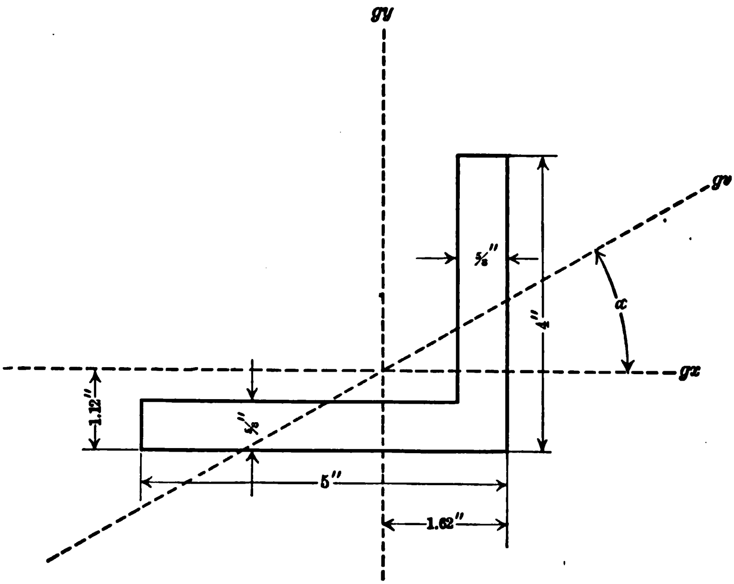


FIG. 119

We shall now determine the moment of inertia of this section with respect to the horizontal and vertical gravity axes, I_{gx} and I_{gy}, and also with respect to an axis v (see Art. 76), making an angle α with the axis x. Consider

the section divided into two rectangles, one $5'' \times \frac{5}{8}''$, which we may call F_1, and the other $3\frac{3}{8}'' \times \frac{5}{8}''$, which we may call F_2. The moment of inertia of the section, with respect to x, is equal to the moment of inertia of F_1 with respect to x plus the moment of inertia of F_2 with respect to x, so that $I_{gx} = \frac{1}{12}(5)(\frac{5}{8})^3 + 5(\frac{5}{8})(.808)^2 + \frac{1}{12}(\frac{27}{8})^3 \frac{5}{8} + \frac{27}{8}(\frac{5}{8})(1.19)^2$ $= 7.14$ in. to the 4th power. Similarly

$I_{gy} = \frac{1}{12}(\frac{27}{8})(\frac{5}{8})^3 + \frac{27}{8}(\frac{5}{8})(1.31)^2 + \frac{1}{12}(\frac{5}{8})(5)^3 + 5(\frac{5}{8})(.88)^2$ $= 12.61$ in. to the 4th power.

NOTE. The problem of finding the moment of inertia of angle sections, channel sections, Z-bar sections, and the built-up sections shown in Figs. 121–125, is of special interest and importance to engineers, occurring as it does in the computation of the strength of all beams and columns made up of these shapes.

Problem 124. Find the moment of inertia of the Z-bar section shown in Fig. 120 for the gravity axes g_x and g_y.

HINT. Divide the area into three rectangles.

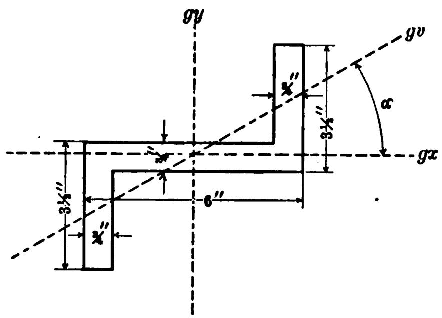


FIG. 120

Problem 125. Compute the moment of inertia for the channel section, shown in Fig. 40, Problem 31, for the horizontal and vertical gravity axes.

Problem 126. Required the moment of inertia of the T-section (Fig. 41, Problem 32), also the moment of inertia of the U-section (Fig. 42, Problem 33) with respect to both horizontal and vertical gravity axes.

Problem 127. The section shown in Fig. 121 consists of a web section and 4 angles, as shown. Find the moment of inertia of the

whole section with respect to the horizontal gravity axis. Given, the moment of inertia of an angle section with respect to its own gravity axis, g', is 28.15 in. to the 4th power.

Problem 128. Consider the section given in Problem 127 to be so taken that it includes two $\frac{7}{8}$-inch rivet holes, as indicated by the

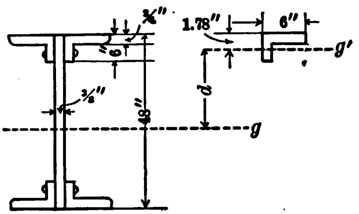


Fig. 121

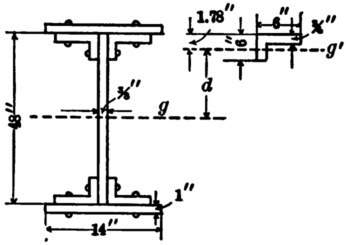


Fig. 122

position of the rivets in Fig. 121. Compute the moment of inertia of the whole section, when the moment of inertia of the rivet holes is deducted. The distance from the center of the rivet hole to the outside of the angle section may be taken as 3 in. Compare the result with that obtained in the succeeding problem.

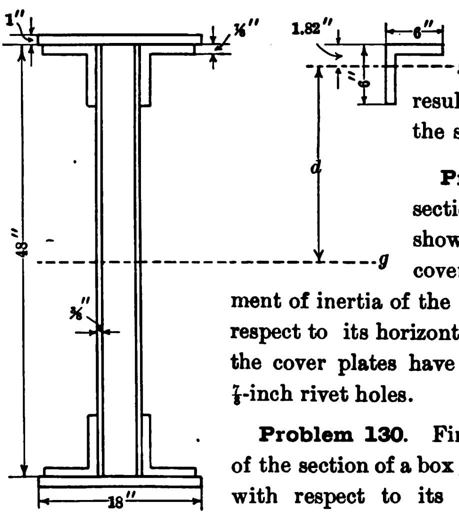


Fig. 123.

Problem 129. The same section shown in Fig. 121 is shown in Fig. 122 with two cover plates. Find the moment of inertia of the whole beam section with respect to its horizontal gravity axis, now that the cover plates have been added. Allow for $\frac{7}{8}$-inch rivet holes.

Problem 130. Find the moment of inertia of the section of a box girder, shown in Fig. 123, with respect to its horizontal gravity axis. The moment of inertia of one of the angle sec-

tions with respect to its own horizontal gravity axis, is 31.92 in. to the 4th power.

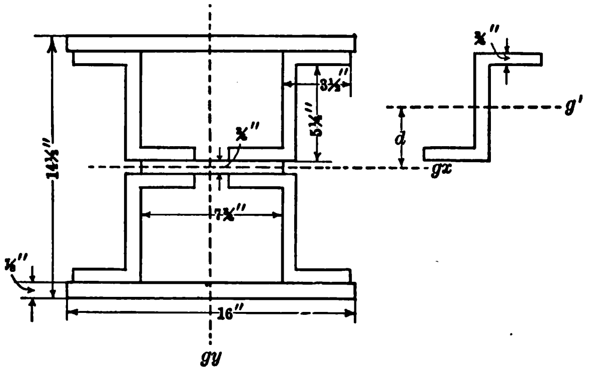


FIG. 124

Problem 131. Find the moment of inertia of the column section, shown in Fig. 124, with respect to the two gravity axes g_x and g_y.

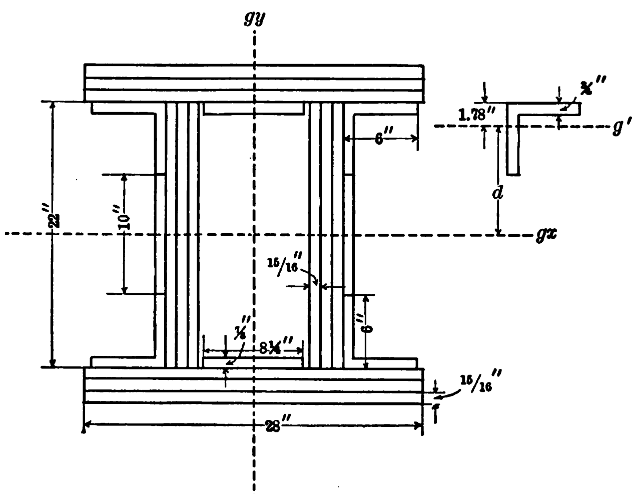


FIG. 125

The column is built up of one central plate, two outside plates, and four Z-bars. The legs of the Z-bars are equal, and have a length of $3\frac{1}{2}$ in. The moments of inertia of each Z-bar section with respect to its own horizontal and vertical gravity axis are 42.12 and 15.44 in. to the 4th power, respectively.

Problem 132. Find the moment of inertia of the section shown in Fig. 125, with respect to the horizontal and vertical gravity axes g_x and g_y. This section is made up of plates and angles. The moment of inertia of each angle section with respect to both its own horizontal and vertical gravity axes is 28.15 in. to the 4th power.

74. Moment of Inertia by Graphical Method. — It will often be necessary to find the moment of inertia of a plane section whose bounding curve is of a complicated form, as

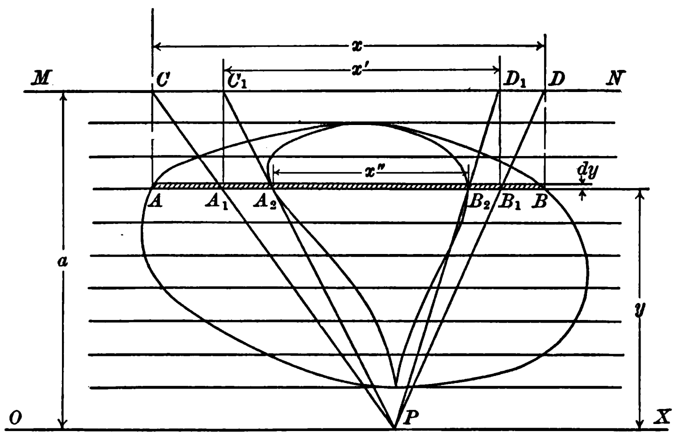


FIG. 126

in the case when it is necessary to compute the strength of rails or deck beams. The graphical method given below may be used for such cases.

Let F be the area bounded by the outer curve of Fig.

126 and let OX be a line with respect to which it is desired to find the moment of inertia of F.

Draw a line MN parallel to OX at a convenient distance, a, from OX. Let AB be any chord of the boundary curve of F parallel to OX. Project AB on MN, obtaining CD. From C and D draw lines to any point P on OX, cutting AB in A_1 and B_1. Project A_1B_1 on MN, obtaining C_1D_1. Join C_1 and D_1 to P by lines cutting AB in A_2 and B_2. If this be done for all positions of AB, a new curve is obtained, as is shown in Fig. 126. Let F'' be the area bounded by this curve.

Letting $AB = x$, $A_1B_1 = x'$, and $A_2B_2 = x''$, it is seen from similar triangles that

$$\frac{x}{x'} = \frac{a}{y} \text{ and } \frac{x'}{x''} = \frac{a}{y},$$

from which

$$\frac{x}{x''} = \frac{a^2}{y^2} \quad \text{or} \quad x = \frac{a^2}{y^2}x''.$$

Then, if $I =$ moment of inertia of F with respect to OX,

$$I = \int y^2 x dy = \int a^2 x'' dy = a^2 \int dF'',$$

or

$$I = a^2 F''.$$

The area F'' may be measured by a planimeter or by one of the methods previously explained (Arts. 40–42).

If it is desired to measure the area of F'' by Simpson's Rule, the area F may first be divided into an even number of strips of the same width by horizontal lines and on the lines thus drawn the points of the auxiliary curve determined by the method described above. The area F'' will then be ready for the application of Simpson's Rule.

If, in laying off the area F, 1 inch parallel to OX represents m inches and 1 inch perpendicular to OX represents n inches, then, measuring a in inches and F'' in square inches, the value I in inches^4 is given by

$$I = a^2F''mn^3.$$

It is left for the student to show that the distance, $\bar{y}$, of the center of gravity of F from OX is given by

$$\bar{y} = \frac{aF'}{F}$$

where F' is the area of the curve traced out by A_1 and B_1 in Fig. 126.

Problem 133. By the above method find the moment of inertia of the area of a circle about a diameter. Determine the area of F'' by the use of Simpson's Rule.

Problem 134. By the above method find the moment of inertia of a rectangle about one side. Take the point P at one of the vertices of the rectangle and show that the boundary curve of F'' is a parabola.

Problem 135. Find the moment of inertia of the area of the rail section of Fig. 68 with respect to the base line.

Using the values of the area and height of the center of gravity found for the figure in Problem 72, find the moment of inertia of the area with respect to a horizontal gravity axis.

75. Moment of Inertia of an Area, (*a*) by Direct Addition, and (*b*) by Use of Simpson's Rule.

(*a*) Divide the area into n narrow strips of equal width Δx parallel to the line OY with respect to which the moment of inertia is desired (Fig. 127). If the middle lengths of these strips are respectively $y_1, y_2, \cdots y_n$,

and their distances from OY respectively $x_1, x_2, \cdots, x_n$, the moment of inertia of the area is given approximately by the formula,

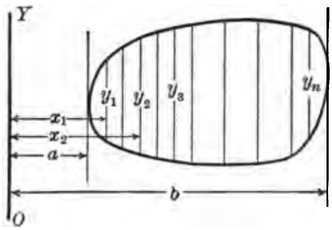


Fig. 127

$$I = x_1^2 A_1 + x_2^2 A_2 + \cdots + x_n^2 A_n,$$

or, since $\Delta x = \dfrac{b-a}{n}$, and, approximately, $A_1 = y_1 \Delta x$, $A_2 = y_2 \Delta x$, $\cdots$ $A_n = y_n \Delta x$, the approximate formula, becomes

$$I = \frac{b-a}{n}[x_1^2 y_1 + x_2^2 y_2 + \cdots + x_n^2 y_n].$$

A more accurate result is usually obtained by the use of Simpson's Rule.

(*b*) Divide the area into an even number of strips of width Δx parallel to the line with respect to which the moment of inertia is sought (Fig. 128).

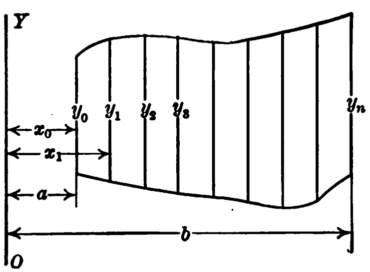


Fig. 128

Since $I = \int_a^b x^2 y\,dx$, where y is the variable length of the strips, it may be evaluated by Simpson's Rule as in Art. 41, the quantity $x^2 y$ taking the place of y in $\int_a^b y\,dx$. Hence, approximately,

$$I = \frac{b-a}{3n}[x_0^2 y_0 + 4(x_1^2 y_1 + x_3^2 y_3 + \cdots + x_{n-1}^2 y_{n-1}) + 2(x_2^2 y_2 + x_4^2 y_4 + \cdots + x_{n-2}^2 y_{n-2}) + x_n^2 y_n].$$

Problem 136. By the direct addition formula compute the moment of inertia of a rectangle 4 in. by 2 in. with respect to a 2 in. side, using (1) 5 divisions, (2) 10 divisions. Compare with the exact value.

Problem 137. Solve the preceding problem by the use of Simpson's Rule, using (1) 6 divisions, (2) 10 divisions.

Problem 138. Find the moment of inertia of the area of the section of Fig. 68 with respect to the base by the method of direct addition.

Problem 139. Solve the preceding problem by the use of Simpson's Rule.

76. Least Moment of Inertia of Area. — In considering the strength of columns and struts it is necessary to know the axis about which the moment of inertia of a cross section is a minimum, since bending will take place about this axis. It was shown in Art. 65 that, if the moment of inertia of the area with respect to two rectangular axes in its plane is known, the moment of inertia with respect to any other axis, making an angle α with one of these, could be found. It was further developed (Art. 67) that the value of α that would render the moment of inertia a minimum was given by the equation

$$\tan 2\alpha = \frac{2\int xy\,dF}{I_y - I_x}.$$

In case either of the axes x or y is an axis of symmetry, the value of α given by this criterion is zero, so that, for areas having an axis of symmetry, the axis of least moment of inertia is the axis of symmetry or the one perpendicular to it.

As an illustration of the problem in general let it be required to find the least moment of inertia of the angle

section shown in Fig. 119 with respect to any axis in the plane of the area through the center of gravity. Let v be the gravity axis making an angle α with the x-axis. The problem then is to find such a value of α that I_{gv} will be a minimum. From Art. 65 we have

$$I_{gv} = I_{gx}\cos^2\alpha - \sin 2\alpha \int xydxdy + I_{gy}\sin^2\alpha.$$

In Art. 73 it was found that $I_{gx} = 7.14$ and $I_{gy} = 12.61$. We proceed now to find the value of $\int xydxdy$ for the angle section. For this purpose, suppose the section composed of two rectangles, F_1 (5 in. × $\frac{5}{8}$ in.), and F_2 ($\frac{27}{8}$ in. × $\frac{5}{8}$ in.), and then find the value of the integral for the two rectangles separately. Considering first the area F_1, and using the double integration, we get

$$\int_{1.62}^{-3.38}\int_{-.495}^{-1.12} xydxdy = \int_{1.62}^{-3.38} xdx\left[\frac{(-1.12)^2}{2} - \frac{(-.495)^2}{2}\right]$$

$$= .505\left[\frac{(-3.38)^2}{2} - \frac{(1.62)^2}{2}\right] = 2.22.$$

In a similar way for F_2, we have

$$\int_{.995}^{1.6}\int_{-.495}^{2.88} xydxdy = \int_{.995}^{1.62} xdx\left[\frac{(2.88)^2}{2} - \frac{(-.495)^2}{2}\right]$$

$$= 4.025\left[\frac{(1.62)^2}{2} - \frac{(.995)^2}{2}\right] = 3.29.$$

Therefore, $\int xydxdy$ for the whole area of the angle section is 5.51 in. to the 4th power. From this we find

$$\tan 2\alpha = \frac{11.02}{5.47} = 2.02.$$

K

Therefore $$2\alpha = 63° \, 40',$$
$$\alpha = 31° \, 50'.$$

The expression for I_{gv} now becomes

$$I_{gv} = 7.14 \cos^2 (31° \, 50') - 5.51 \sin (63° \, 40') + 12.61 \sin^2 (31° \, 50') = 3.72 \text{ in. to the 4th power.}$$

This gives the least radius of gyration,

$$k_{gv} = .84 \text{ in.}$$

Problem 140. Find the least moment of inertia I_{gv} and least radius of gyration k_{gv} of the Z-section shown in Fig. 120. In this case $I_{gx} = 15.44$ in. to the 4th power and $I_{gy} = 42.12$ in. to the 4th power.

Ans. Least $I_{gv} = 5.61$ in. to 4th power and least $k_{gv} = .80$ in.

Problem 141. An angle iron has equal legs. The section, similar to that in Fig. 119, is 8 in. × 8 in. with a thickness of ½ in. Find I_{gx}, I_{gy}, least I_{gv}, and least k_{gv}.

Ans. $I_{gx} = I_{gy} = 48.58$ in. to the 4th power,
$I_{gv} = 19.55$ in. to the 4th power.
$k_{gv} = 1.59$ in.

Problem 142. Find the moment of inertia of the column section, shown in Fig. 124, with respect to an axis v making an angle of 30° with gx. What value of α makes I_{gv} minimum in this case?

77. The Ellipse of Inertia.—It is interesting to note, at this point, the relations between the moments of inertia with respect to all the lines, in the plane of the area, passing through a point. We have seen that for every point in an area there is always a pair of rectangular axes for one of which the moment of inertia is a maximum and for the other a minimum; that is, there is always a pair of principal axes. The criterion for such axes was found to be

$$\tan 2\alpha = \frac{2i}{I_y - I_x},$$

which means, since the tangent of an angle may have any value from 0 to infinity, positive and negative, that for every point there is always a pair of axes such that

$$i = 0, \text{ i.e. } \int xy\,dF = 0.$$

This means that the expression for I_v may always be reduced to the form

$$I_v = I_x \cos^2 \alpha + I_y \sin^2 \alpha \qquad \text{(Art. 65)}$$

by properly selecting the axes of reference, where now I_x and I_y represent the principal moments of inertia. If we divide through by F, the equation becomes

$$k_v^2 = k_x^2 \cos^2 \alpha + k_y^2 \sin^2 \alpha.$$

Let $$\rho = \frac{1}{k_v}, \quad a = \frac{1}{k_x}, \quad \text{and } b = \frac{1}{k_y},$$

then $$\frac{1}{\rho^2} = \frac{\cos^2 \alpha}{a^2} + \frac{\sin^2 \alpha}{b^2},$$

or $$1 = \frac{\rho^2 \cos^2 \alpha}{a^2} + \frac{\rho^2 \sin^2 \alpha}{b^2},$$

which is the equation of an ellipse referred to the principal axes of inertia as axes, the coördinates of any point on the ellipse being $x = \rho \cos \alpha$, $y = \rho \sin \alpha$.

Hence if OX and OY are respectively the axes of maximum and minimum moments of inertia of an area for a given point O, there exists an ellipse with minor and major axes on OX and OY respectively such that the distance from O to the ellipse along any line is the reciprocal of the radius of gyration of the area with respect to that

line. This ellipse is called the *ellipse of inertia* of the area for the given point (Fig. 129).

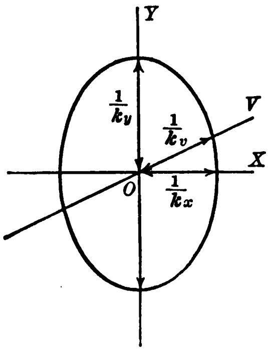


FIG. 129

The ellipse of inertia furnishes a graphical method of finding the moment of inertia of an area for any axis through a point when the principal moments of inertia of the area for that point are known.

Problem 143. Sketch the ellipse of inertia of the area of a rectangle for (*a*) the intersection of the diagonals, (*b*) the middle point of one side of the rectangle.

Problem 144. Determine the principal moments of inertia of the area of a rectangle 6 inches by 10 inches for axes passing through a vertex of the rectangle. Sketch the ellipse of inertia.

Problem 145. Determine the principal moments of inertia of the area of a circle for a point on its circumference, and sketch the ellipse of inertia of the area for that point.

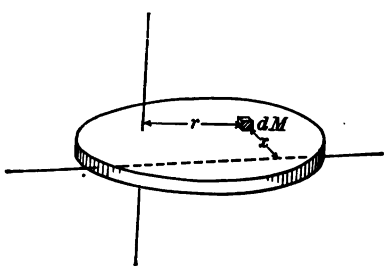


FIG. 130

78. Moment of Inertia of a Thin Plate, (*a*) with Respect to an Axis in the Plate Parallel to its Faces, (*b*) with Respect to an Axis Perpendicular to its Faces. —(*a*) Suppose the plate to be of constant small thickness t and unit weight γ and let x be the distance of an element of mass dM from the axis, where $dM = \frac{\gamma}{g} t dF$ (Fig. 130).

Then approximately

$$I = \int x^2 dM = \frac{\gamma}{g} t \int x^2 dF.$$

But $\int x^2 dF$ is the moment of inertia of the area of the face of the plate about the given axis. Therefore, approximately, *the moment of inertia of a thin plate with reference to an axis of the plate parallel to its faces equals* $\frac{\gamma t}{g}$ *times the moment of inertia of the area of its face with reference to the same axis.*

(*b*) Let r be the distance of dM from the axis perpendicular to the face of the plate (Fig. 130). Then the moment of inertia of the plate with respect to the axis is

$$I = \int r^2 dM = \frac{\gamma t}{g} \int r^2 dF.$$

Or, approximately, *the moment of inertia of a thin plate with reference to an axis perpendicular to the faces of the plate equals* $\frac{\gamma t}{g}$ *times the moment of inertia of the face of the plate with reference to that axis.*

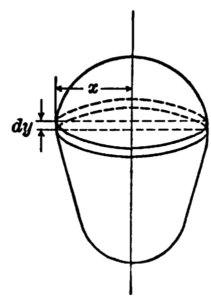


FIG. 131

79. Moment of Inertia of Solid of Revolution. — Consider the moment of inertia of a solid of revolution with respect to its axis of revolution. Imagine the solid cut into thin slices, all of the same thickness, by parallel planes perpendicular to the axis of revolution (Fig. 131). Each slice is a circular disk of thickness dy and radius x, and its

moment of inertia with respect to the axis of revolution is $\frac{\gamma}{g} dy \pi x^2 \cdot \frac{x^2}{2}$ (Art. 71). The moment of inertia of the solid of revolution is the sum of the moments of the small slices, so that

$$I_{gr} = \int \frac{\gamma \pi}{2g} x^4 dy,$$

the limits of integration and the relation between x and y depending upon the particular solid considered.

Problem 146. Prove that the moment of inertia of a right circular cylinder of radius r and altitude h with respect to its axis is $\frac{Mr^2}{2}$.

Problem 147. Prove that the moment of inertia of a right circular cone with respect to its axis is $I_{gh} = \frac{3}{10} Mr^2$.

Problem 148. Prove that the moment of inertia of a sphere with respect to a diameter is $\frac{2}{5} Mr^2$.

Problem 149. The surface of a spheroid is generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x-axis. Prove that the moment of inertia of the solid inclosed with respect to the axis of revolution is $I_{gx} = \frac{2}{5} Mb^2$.

Problem 150. Prove that the moment of inertia of a rectangular parallelopiped with edges a, b, and c, with respect to a gravity line parallel to the edge c is $I_{gc} = \frac{M}{12}(a^2 + b^2)$.

Problem 151. Prove that the moment of inertia of a slender rod of length a with respect to an axis perpendicular to the rod and passing (*a*) through the center, (*b*) through the end, is (*a*) $\frac{Ma^2}{12}$, (*b*) $\frac{Ma^2}{3}$.

Problem 152. An anchor ring is generated by revolving a circle of radius a about a line in its plane distant b from the center of the circle. Show that the moment of inertia of the mass of the anchor ring with respect to the axis of revolution is $M(b^2 + \frac{3}{4} a^2)$.

80. Moment of Inertia of a Body by Parallel Sections. — By dividing a body up into thin plates by parallel planes, parallel to the axis with respect to which the moment of inertia of the body is sought, the moment of inertia is made to depend upon the moments of inertia of the areas of the sections and their distances from the given axis.

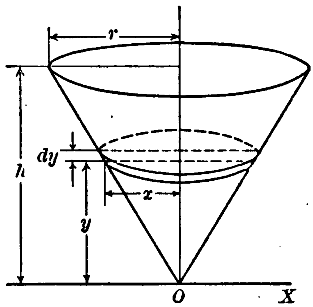


FIG. 132

As an illustration consider the problem of finding the moment of inertia of a right circular cone with respect to an axis through the vertex perpendicular to the axis of the cone (Fig. 132). The moment of inertia of the thin plate with respect to the diameter of its lower face is approximately $\frac{\gamma dy}{g} \cdot \frac{\pi x^4}{4}$ (Arts. 78 and 71). The moment of inertia of this plate with respect to OX is therefore

$$\frac{\gamma dy}{g}\frac{\pi x^4}{4} + \frac{\gamma \pi x^2}{g} dy \cdot y^2,$$

or

$$\frac{\pi\gamma}{g}\left(\frac{x^4}{4} + x^2y^2\right)dy,$$

and the I_x for the cone is then

$$I_x = \frac{\pi\gamma}{g}\int_0^h\left(\frac{x^4}{4} + x^2y^2\right)dy.$$

From the figure x may be evaluated in terms of y and the integral evaluated.

Problem 153. Prove that the moment of inertia of a right circular cone with respect to an axis through the vertex perpendicular to the axis of the cone is $\frac{3}{20} M(r^2 + 4h^2)$.

Problem 154. Using the result of Problem 153, find the moment of inertia of the cone about a gravity axis parallel to the base, and then about a diameter of the base. $I_{gx} = \frac{3}{20} M\left(r^2 + \frac{h^2}{4}\right)$.

Problem 155. Prove that the moment of inertia of a right circular cylinder with respect to a diameter of the base is $M\left(\frac{r^2}{4} + \frac{h^2}{3}\right)$.

Problem 156. Prove that the moment of inertia of an elliptical right cylinder of height h and semi-axes of base a and b with respect to the diameter $2a$ of the base is $M\left(\frac{b^2}{4} + \frac{h^2}{3}\right)$, and with respect to a gravity line parallel to the diameter $2a$ is $\frac{M}{12}(3b^2 + h^2)$.

Problem 157. Show that the moment of inertia of a right circular cylinder, altitude h, and radius of base r, with respect to a gravity axis parallel to the base is $I_{gx} = M\left(\frac{r^2}{4} + \frac{h^2}{12}\right)$, and find the moment of inertia with respect to an axis parallel to this and at a distance d from the base.

Problem 158. It is required to find the moment of inertia of the cast-iron disk flywheel shown in Fig. 133 with respect to its geometrical axis.

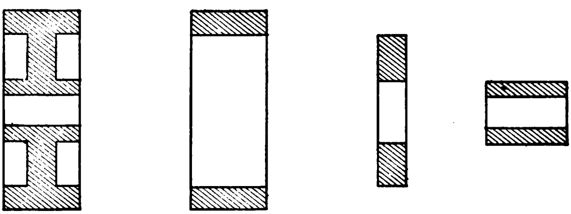


FIG. 133

HINT. The wheel may be regarded as made up of three hollow cylinders, the moment of inertia of the whole wheel being equal to the sum of the moments of inertia of the three parts. The dimensions are as follows: diameter of wheel 2 ft., width of rim and hub 4 in., thickness of rim and web 2 in., thickness of hub 1¼ in., and diameter of shaft 2 in. All distances must be in feet.

Problem 159. Find the moment of inertia of the cast-iron fly-wheel shown in Fig. 134 with respect to its axis of rotation. There are six elliptical spokes, and these may be regarded as of the same cross section throughout their entire length.

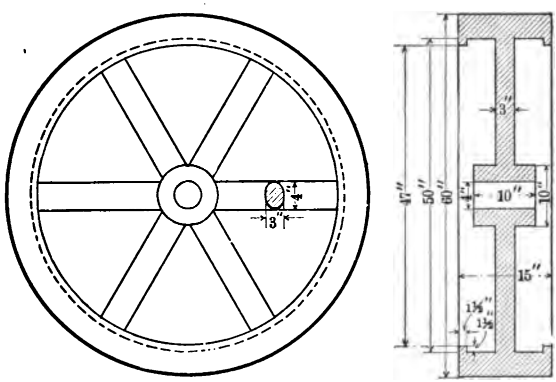


FIG. 134

Problem 160. Find the mass and moment of inertia, with respect to the axis, of a right circular cylinder whose density varies directly as the distance from the axis and is γ_1 at the outside of the cylinder.

$$Ans. \quad M = \frac{2\,\gamma_1 \pi r^2 h}{3\,g}, \qquad I = \frac{3}{5} M r^2.$$

81. Moment of Inertia of a Mass; Inclined Axis. — We shall now study the problem of finding the moment of inertia of a solid with respect to an axis inclined to the

coördinate axes. Suppose the moments of inertia of the body with respect to the three coördinate axes known from the expressions:

$$I_x = \int (z^2 + y^2)\,dM,$$

$$I_y = \int (z^2 + x^2)\,dM,$$

$$I_z = \int (y^2 + x^2)\,dM,$$

and let it be required to find the moment of inertia of the body with respect to any other axis OV making angles

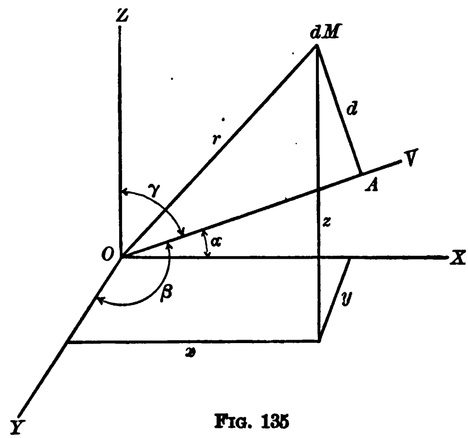


FIG. 135

α, β, γ with the coördinate axes. (See Fig. 135.) Let dM equal the mass of an infinitesimal portion of the body and d its distance from the axis OV.

Since $r^2 = x^2 + y^2 + z^2$, $OA = x \cos\alpha + y \cos\beta + z \cos\gamma$ and

$$d^2 = r^2 - \overline{OA}^2 = (x^2 + y^2 + z^2) - (x \cos\alpha + y \cos\beta + z \cos\gamma)^2$$

we may write

$$
\begin{aligned}
I_v &= \int d^2 dM \\
&= \int [(x^2 + y^2 + z^2) - (x \cos \alpha + y \cos \beta + z \cos \gamma)^2] dM.
\end{aligned}
$$

This reduces, since $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$, to

$$
\begin{aligned}
I_v = \int (y^2 + z^2) \cos^2 \alpha dM &+ \int (z^2 + x^2) \cos^2 \beta dM \\
&+ \int (x^2 + y^2) \cos^2 \gamma dM - 2 \cos \alpha \cos \beta \int xy dM \\
&- 2 \cos \beta \cos \gamma \int yz dM - 2 \cos \gamma \cos \alpha \int xz dM,
\end{aligned}
$$

or

$$
\begin{aligned}
I_v = I_x \cos^2 \alpha + I_y \cos^2 \beta + I_z \cos^2 \gamma &- 2 \cos \alpha \cos \beta \int xy dM \\
&- 2 \cos \beta \cos \gamma \int yz dM - 2 \cos \gamma \cos \alpha \int xz dM,
\end{aligned}
$$

which gives the moment of inertia of the body with respect to an inclined axis in terms of the moments of inertia with respect to the coördinate axes and the products of inertia $\int xy dM$, $\int yz dM$, and $\int xz dM$.

82. Principal Axes. — If the three products of inertia $\int xy dM$, $\int yz dM$, and $\int xz dM$ are each equal to zero, the expression for I_v reduces to the form

$$I_v = I_x \cos^2 \alpha + I_y \cos^2 \beta + I_z \cos^2 \gamma.$$

In this case the coördinate axes x, y, and z are called the *principal axes* for the point O and the moments I_x, I_y, and I_z, the *principal moments of inertia* for that point.

If the point O is the center of gravity of the body, the principal axes are called the *principal axes of the body*. It can be shown that it is always possible to select the coördinate axes x, y, and z so that the products of inertia given in the expression for I_v will each be zero. It follows that for every point of a body there exists a set of rectangular axes that are principal axes.

If the xy-plane is a plane of symmetry, the products of inertia $\int zydM$ and $\int zxdM$ are both zero, for to any term in $\Sigma zydM$, as z_1y_1dM, there corresponds a term, $-z_1y_1dM$, equal in numerical value but opposite in sign. Hence if two of the coördinate planes are planes of symmetry of a body, the three products of inertia with respect to these planes are zero, and the coördinate axes are the principal axes of the body through the origin.

Problem 161. What are the principal axes of a sphere through a point on its surface? What are the values of the principal moments of inertia for that point? $\frac{2}{5}Mr^2$, $\frac{7}{5}Mr^2$, $\frac{7}{5}Mr^2$.

Problem 162. Find the moment of inertia of the ellipsoid whose surface is given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

with respect to the axes a, b, and c, and with respect to an inclined axis OV making angles α, β, γ with a, b, and c, respectively. The volume of an ellipsoid is

$$\frac{4\,\pi abc}{3},\quad I_a=\frac{M}{5}(b^2+c^2),\quad I_b=\frac{M}{5}(c^2+a^2),\quad I_c=\frac{M}{5}(a^2+b^2)$$

and $$I_v=I_a\cos^2\alpha+I_b\cos^2\beta+I_c\cos^2\gamma.$$

83. Ellipsoid of Inertia. — It is always possible to reduce the expression for I_v to the form

$$I_v = I_x \cos^2 \alpha + I_y \cos^2 \beta + I_z \cos^2 \gamma,$$

by selecting the axes x, y, and z so that the products of inertia are zero.

Dividing this equation through by M, we have

$$k_v^2 = k_x^2 \cos^2 \alpha + k_y^2 \cos^2 \beta + k_z^2 \cos^2 \gamma.$$

Let $\rho = \frac{1}{k_v}$, $a = \frac{1}{k_x}$, $b = \frac{1}{k_y}$, and $c = \frac{1}{k_z}$. Then the equation becomes, on multiplying by ρ^2,

$$\frac{\rho^2 \cos^2 \alpha}{a^2} + \frac{\rho^2 \cos^2 \beta}{b^2} + \frac{\rho^2 \cos^2 \gamma}{c^2} = 1,$$

which is the equation of an ellipsoid of semi-axes a, b, and c, on the principal axes, the coördinates of any point on the surface being

$$x = \rho \cos \alpha, \quad y = \rho \cos \beta, \quad z = \rho \cos \gamma.$$

Hence for any body there exists for each point an ellipsoid such that the distance from the given point to the surface along any line is the reciprocal of the radius of gyration of the body with respect to that line, the axes of the ellipsoid coinciding with the principal axes for that point.

Since one of the semi-axes, a, b, c, of the ellipsoid is a maximum value and another a minimum value of the distance from the center to points on the surface, it follows that two of the principal moments of inertia of a body for any point are respectively minimum and maximum moments of inertia of the body with respect to lines through the point.

Problem 163. Write the equation and construct the inertia ellipsoid for the center of gravity of a right circular cylinder, altitude h and radius r.

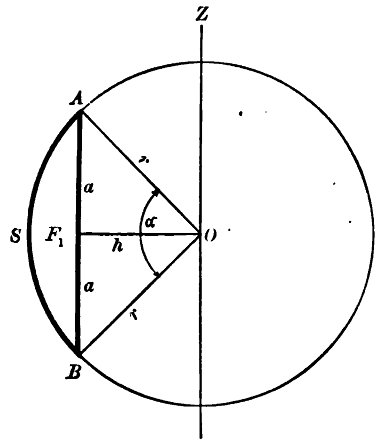


FIG. 136

Problem 164. Construct the inertia ellipsoid for the center of a solid sphere of radius r.

Problem 165. Show that the moment of inertia of the segment of the circle F_1 (Fig. 136) with respect to the axis OZ is

$$\frac{r^4}{4}\left(\frac{\alpha}{2}+\frac{1}{2}\sin\alpha\right),$$

the moment of inertia of the sector $OBSA$, minus $\frac{1}{2}ah^3$, the moment of inertia of the triangle OAB, or

$$I_{OZ}=\frac{r^4}{16}\left(2\alpha-\sin 2\alpha\right);$$

and the moment of inertia of F_1 with respect to OS is $\frac{r^4}{4}\left(\frac{\alpha}{2}-\frac{1}{2}\sin\alpha\right)$, the moment of inertia of the sector, minus $\frac{1}{6}ha^3$, the moment of inertia of the triangle AOB, or $I_{OS}=\frac{r^4}{8}\left(\alpha-\frac{4}{3}\sin\alpha+\frac{1}{6}\sin 2\alpha\right)$.

Problem 166. Show that the moment of inertia of the counterbalance (Fig. 58), with respect to a line through O, perpendicular to OS, and in the plane of the wheel, is approximately

$$I_{OZ}=\left[\frac{r^4}{16}\left(2\alpha-\sin 2\alpha\right)-\frac{r_1^4}{16}\left(2\beta-\sin 2\beta\right)+\frac{4a\,(OO')}{3}-F_2(OO')^2\right]\frac{\gamma t}{g},$$

where $F_2=\frac{\beta r_1^2}{2}-ar_1\cos\frac{\beta}{2}$, $OO'=r_1\cos\frac{\beta}{2}-r\cos\frac{\alpha}{2}$, and t is the thickness, as explained in Art. 38.

Problem 167. Find the moment of inertia of the counterbalance (Fig. 58), with respect to a line through O perpendicular to the plane

of the wheel. It may be written

$$I_O = \left[\frac{r^4}{8}\left(2\alpha - \frac{1}{3}\sin 2\alpha - \frac{4}{3}\sin\alpha\right) - \frac{r_1^4}{8}\left(2\beta - \frac{1}{3}\sin 2\beta - \frac{4}{3}\sin\beta\right) + \frac{4a^3OO'}{3} - F_2(OO')^2\right]\frac{\gamma t}{g}.$$

84. Moment of Inertia of Locomotive Drive Wheel. — The drive wheel may be represented as in Fig. 137, and may be considered as made up of a tire, rim, twenty elliptical spokes, counterbalance, and equivalent weight on opposite side of center, and hub.

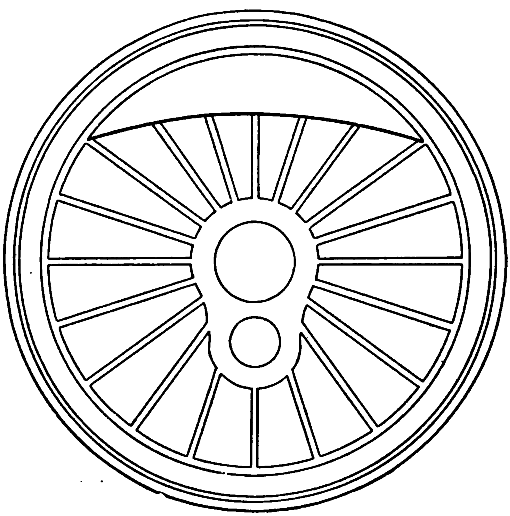


FIG. 137

The dimensions of the wheel are as follows:
Tire, outside radius 40″, inside radius 36″, width 5″.
Rim, outside radius 36″, inside radius 34″, width $4\frac{1}{8}$″.
Hub, outside radius 10″, inside radius $4\frac{3}{4}$″, thickness 8″.

Counterbalance, outside radius 34″, inside radius 7′ 11.5″, thickness $7\frac{1}{4}$″.

20 spokes, 24″ long, elliptical $3\frac{1}{4}$″ by $2\frac{1}{4}$″.

Angle at center subtended by counterbalance, $\alpha = 94° 40'$.

Radius of crank-pin circle 18″.

$\gamma = 490$ lb. per cu. ft.

From the above data the following additional values are computed (Art. 38):

Angle subtended by counterbalance at center of inner boundary circle, $\beta = 30° 20'$.

Mass of counterbalance 17.1.

Distance of center of gravity of counterbalance from center of wheel 28.8″.

$$\text{Mass of weights carried by crank-pin} = \frac{28.8}{18} \times 17.1 = 27.4.$$

(Since moment of counterbalance = moment of crank-pin weights.)

The moment of inertia of the wheel with respect to its axis of rotation is first found. The tire, rim, and hub are hollow cylinders, and their moments of inertia are computed from the formula

$$\frac{\pi}{2}(r_1^4 - r_2^4)\frac{\gamma t}{g}.$$

For the counterbalance the formula of Problem 167 is used. The spokes are regarded as elliptical cylinders with the short axis of the ellipse in the plane of the wheel. The moment of inertia of one spoke is computed from the formula

$$\frac{\pi ab\gamma}{g}\int_{\frac{5}{6}}^{\frac{17}{6}}\left(x^2 + \frac{b^2}{4}\right)dx, \quad \text{(Arts. 77, 72, 64.)}$$

and 10 per cent is deducted for the parts of the spokes included in the counterbalance and boss.

The moment of inertia of the weights at the crank-pin is computed on the assumption that the whole weight is concentrated at the center of the crank-pin.

The results obtained are the following:

Tire	$I_0 = 423$
Rim	136
Hub	7
Spokes	81
Counterbalance	120
Weights at crank-pin	62
Total	$I_0 = 829$

With respect to a gravity axis OZ (Fig. 136) in the plane of the wheel, when the counterbalance is in a position where the line joining its center of gravity to the center is perpendicular to OZ, we get for the moments of inertia of the various parts:

Tire	$I_{OZ} = 212$
Rim	68
Hub	4
Spokes	41
Counterbalance	103
Weights at crank-pin	62
Total	$I_{OZ} = 490$

Since the widths of tire and rim are small compared to the diameter of the wheel, their moments of inertia are closely approximate to the values they would have if the

material of tire and rim were in one plane, which would be one half of the corresponding I_O. The same is true, less accurately, of spokes and hub. The values of I_O have been divided by 2 for the corresponding values of I_{OZ} for the four parts mentioned. The weights at the crank-pin center are at the same distance from OZ as from O, and the moment of inertia is therefore the same as for O. The moment of inertia of the counterbalance was computed from the formula of Problem 166.

Problem 168. Compute the moment of inertia of a pair of drivers and their axle with respect to their axis of rotation. Use the data given above and assume the axle as cylindrical, the diameter being $9\frac{1}{2}''$ and the length $68''$. *Ans.* 1661.

Problem 169. Compute the moment of inertia of the pair of drivers and their axle, given in the preceding problem, with respect to an axis midway between the wheels and perpendicular to the axle. Consider the counterbalance of both wheels in such a position as to give a maximum moment of inertia and the distance between the centers of the wheels $60''$.

Problem 170. Find the moment of inertia of two cast-iron car wheels and their connecting steel axle with respect to (*a*) their axis of rotation, (*b*) an axis midway between the wheels and perpendicular to the axle. Consider the car wheels as composed of an outside tread, a circular web, and a hub; each part may be considered a hollow cylinder with the following dimensions: tread, outside radius $16''$, inside radius $14''$, width $5\frac{1}{2}''$; web, outside radius $14''$, inside radius $5\frac{1}{2}''$, thickness $1.5''$; hub, outside radius $5\frac{1}{2}''$, inside radius $2\frac{1}{2}''$, width $8''$; axle (considered cylindrical), $5''$ diameter and $7'\ 3''$ long. Distance between centers of wheels $60''$. According to the assumption made above, the flange has been neglected, the web is considered a hollow disk, and the axle of uniform diameter throughout its length.

Problem 171. The value 490 is the greatest value for the moment of inertia of a drive wheel with respect to a gravity axis in its plane. The least value will be with respect to an axis at right angles to this through the centers of gravity of the counterbalance and wheel. The student should compute this least moment of inertia.

Problem 172. In Problem 169 the drivers have been considered as having their cranks in the same plane. In practice they are 90° apart. Find the moment of inertia with respect to the axis stated when the wheels are so placed.

CHAPTER VIII

FLEXIBLE CORDS. THE ARCH. BENDING MOMENTS

85. Introduction.—A cord under tension due to any load may be considered as a rigid body. In the analysis of problems in which such cords are considered, the method of cutting or section may be used. Since the cord is flexible (requiring no force to bend it), it is easy to see that, no matter what forces are acting upon it, it must have at any point the direction of the resultant force at that point, and so must be under simple tension. If the cord is curved, as is the case where it is wrapped around a pulley, the resultant force is in the direction of the tangent.

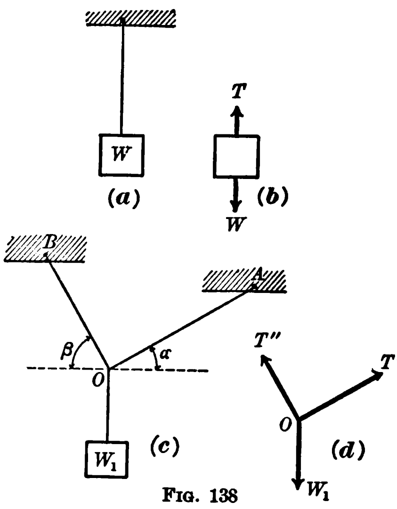


FIG. 138

Consider, as the simplest case, a weight W suspended by a cord, as shown in Fig. 138 (a). The forces acting on W are shown in (b) of the same figure. The cord has been considered cut and the force T, acting vertically

upward, has been used to represent the tension. Summation of vertical forces $= 0$ gives $T = W$. In Fig. 138 (*c*) the weight W_1 is supported by two cords. The system of forces acting on the point O is shown in (*d*), where T' and T'' represent the tensions in the cords A and B respectively. $\Sigma X = 0$ and $\Sigma Y = 0$ give

$$T' \cos \alpha = T'' \cos \beta$$

and

$$T' \sin \alpha + T'' \sin \beta = W_1.$$

These two equations are sufficient to determine the unknown tensions T' and T''.

Problem 173. A weight of 500 lb. is attached to the ends of two cords of length 8 ft. and 6 ft., the other ends of the cords being attached to the points A and B respectively, where A is lower than B, and is distant 9 ft. horizontally and 4 ft. vertically from B. Find the tensions in the cords. Solve analytically and graphically.

Suggestion. Make use of the horizontal and vertical projections of the cords to determine the angles which the cords make with the horizontal.

Problem 174. A weight of 275 lb. is knotted at a point C to a rope which passes over two smooth pulleys at A and B distant 50 ft. apart and in the same horizontal line, carrying weights of 350 lb. and 300 lb. respectively, as in Fig. 139. Find for what position of C the weights will be in equilibrium. Solve analytically and graphically.

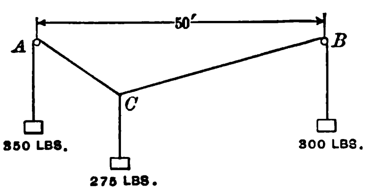


Fig. 139

Problem 175. If, in the preceding problem, B is 10 ft. lower than A and distant 50 ft. horizontally from A, find the position of C for equilibrium. Solve analytically and graphically.

86. Several Suspended Weights. Analytical Method of Solution. — If two weights W_1 and W_2 are attached to the cord, as shown in the case of the cord $ABCD$ (Fig. 140), each portion is under tension. Consider the cord cut at A and D and represent the tensions by T_1 and T_2 respectively. From $\Sigma X = 0$ and $\Sigma Y = 0$ we have

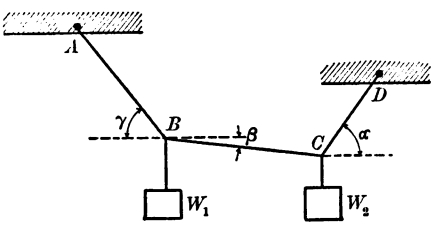


Fig. 140

$$T_1 \cos \gamma = T_2 \cos \alpha$$

and $$T_1 \sin \gamma + T_2 \sin \alpha = W_1 + W_2.$$

A consideration of the forces acting at B, if we call the tension in the portion BC, T_3, gives, when the summation of the x and y forces are each put equal to zero,

$$T_1 \cos \gamma = T_3 \cos \beta$$

and $$T_1 \sin \gamma - T_3 \sin \beta = W_1.$$

In a similar way, consider the forces acting on the point C, and we have

$$T_3 \cos \beta = T_2 \cos \alpha$$

and $$T_3 \sin \beta + T_2 \sin \alpha = W_2.$$

Of the six equations given above only four are independent; consequently, of the six quantities T_1, T_2, T_3, α, β, and γ, two must be known in order to determine the other four, or else additional conditions must be given for determining two more independent equations. Such conditions may, for example, be the lengths of the cords and

the positions of the two points of support, A and D, of Fig. 140.

In general, if there are n knots such as B and C of Fig. 140, with the weights W_1, W_2, W_3, W_4, etc., attached, it will be possible to get $n+2$ independent equations. These will be sufficient to determine the tension in each portion of the cord and its direction, provided the tension at A, say, and its direction are known. If the weights are close together, the curve takes more nearly the form of a smooth curve. Two special cases of this kind are discussed in this chapter in Art. 97 and Art. 100.

Problem 176. The points A and D of Fig. 140 are distant 20 ft., apart, and are in the same horizontal line. Each of the cords is 10 ft. long and the weights are each 50 lb. Find the tensions in the cords.

Problem 177. In Fig. 140 $W_1 = 50$ lb., $W_2 = 120$ lb., $AB = 10$ ft., $\gamma = 60°$. D and A are on the same level, 20 ft. apart. Find the tensions in CD and BC and the lengths of these cords.

87. Graphical Method of Solution.—In Fig. 141 (*i*) is shown a cord carrying the weights W_1, W_2, W_3, and in (*ii*) the corresponding stresses in the cord and the reaction at the supports. The cord in this position may be thought of as a rigid body in equilibrium under the action of the weights and the reactions of the supports. The weights and the reactions of the supports therefore form a system of forces in equilibrium and the relation found in Art. 57 must exist here between the rays to the force polygon and the sides of the equilibrium polygon.

In stating the problem for solution enough conditions must be given to determine the construction. If the

points of support are given and assigned weights are to act in assigned vertical lines, then there exist an indefinite number of solutions. For the tensions in the strings can be increased or decreased by shortening or lengthening them.

If the string were cut at any point and held in position, the part of the string to one side of the cut would be in

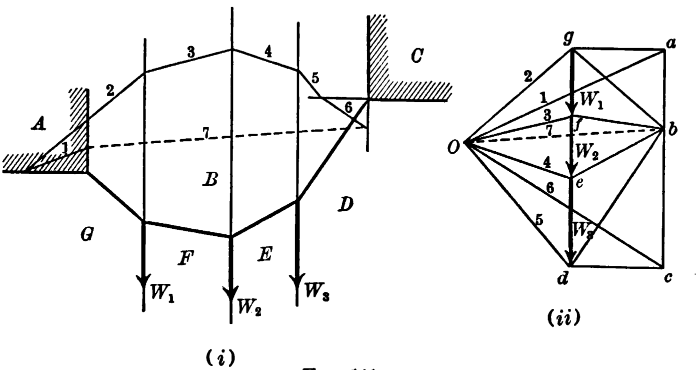


FIG. 141

equilibrium under the action of the forces acting upon that part. The horizontal component of the tension at the point where the cut is made must then be equal to the horizontal component of the reaction at either support. The horizontal component of the tension of the string is therefore the same at all points and is equal to the horizontal component of either reaction.

If, now, in addition to having given the points of support, the weights and their lines of action, there is also given the horizontal component of the tension in the string, the graphical construction for the shape of the

string and the tensions in the segments can be carried out. The force polygon *agf*$\cdots$*ca* is laid off, the point *b* not being determined. Any point *O* is chosen and the rays *oa*, *og*, $\cdots$ *oc* are drawn. Horizontal and vertical lines are drawn through the points of support as action lines of the horizontal and vertical components of the reactions at these points. Beginning at any convenient point, as on *AB*, the equilibrium polygon is drawn with sides respectively parallel to the rays of the force polygon. Parallel to the closing side (dotted) of the equilibrium polygon, a ray (dotted) is drawn from *O* to *ca* determining the point *b*. Therefore *cb* and *ba* are respectively the vertical components of the reactions of the right and left supports with the given value of the horizontal component of the tension. The triangle *bag* is then the triangle of forces acting at the left support, and *bg* therefore represents the tension in the segment of the cord attached there. The segment of the cord *BG* may then be drawn parallel to *bg* from the point of support to the vertical line in which W_1 acts. In the same way the triangles *bgf*, *bfe*, and *bed* are the force triangles for the points where the weights are attached and *bf*, *be*, and *bd* represent the tensions in the segments *BF*, *BE*, and *BD* respectively. They give therefore the direction of these segments. The shape of the cord can then be constructed.

A check on the accuracy will be that the last segment must pass through the right support.

The segments of the cord form an equilibrium polygon, or *string polygon*, for which the corresponding ray polygon has *b* as a pole.

Problem 178. Weights of 25, 40, and 50 lb. are to be suspended by a cord in lines distant 4, 8, and 11 ft. respectively from the left support. The right support is to be distant 15 ft. horizontally and 3 ft. higher than the left. With a horizontal component of tension in the cord equal to 30 lb., find by graphical methods the shape of the cord, the lengths of the segments, the tensions in the segments, and the reactions at the support.

Problem 179. Ten equal weights are to be hung at equal horizontal distances on a cord supported at two points on the same level. The horizontal tension in the cord is to equal five times the value of one weight. Draw the shape of the string and determine the maximum tension. Draw on the same figure the shape of the cord when the horizontal tension is three times the value of one weight.

88. Locus of the Pole of the String Polygon.—Let the left- and right-hand vertical components of the reactions at the supports be Y_1 and Y_2 respectively, H the horizontal component of the tension, a_1, a_2, a_3 the horizontal distances of the weights W_1, W_2, W_3 from the left support, l the horizontal distance between the supports, and h the vertical height of the right support above the left.

Taking moments about the left support,

$$Y_2 = \frac{a_1 W_1 + a_2 W_2 + a_3 W_3}{l} + \frac{h}{l} H.$$

Let $$Y_2' = \frac{a_1 W_1 + a_2 W_2 + a_3 W_3}{l}.$$

Then $$Y_2 = Y_2' + \frac{h}{l} H.$$

The value of Y_2' is the value that Y_2 would have if either h or H were zero, *i.e.* if the supports were on the same level, or if the string were replaced by a rigid body simply resting on the supports without horizontal pressure.

The value Y_2' may be computed analytically and laid off vertically from d on the load line gd (Fig. 142), or it may be located by using the equilibrium polygon for vertical

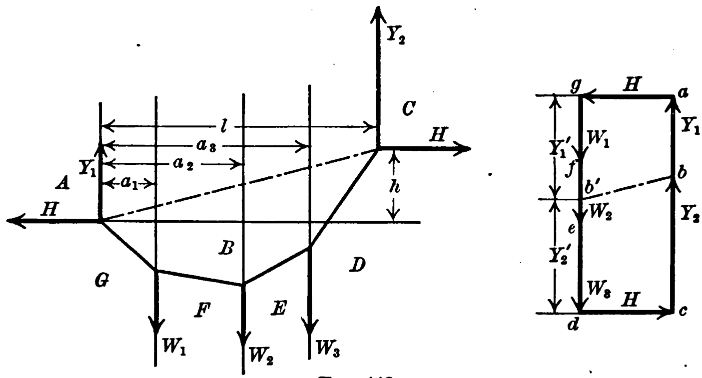


FIG. 142

forces only, assuming H to be zero. Y_2 is then found by increasing Y_2' by the quantity $\frac{h}{l}H$, which is proportional to H. If through the end, b', of Y_2' a line is drawn parallel to the line joining the points of support, the pole, b, of the string polygon is found on this line at a horizontal distance H from the load line gd. Conversely, any point on this line may be taken as a pole for a string polygon, and the corresponding value of H is the horizontal distance from the point chosen to the load line gd. The locus of the poles of the string polygons is therefore the straight line parallel to the line joining the points of support and passing through the points which would divide the load line into the reactions if the supports were on the same level.

By varying the position of the pole the form of the string polygon may be changed.

89. Pole Distance and Depth of String Polygon. — *The depth of the string polygon, measured from the line joining the points of support, varies inversely as the pole distance from the load line.* For, if d is the depth of the string polygon

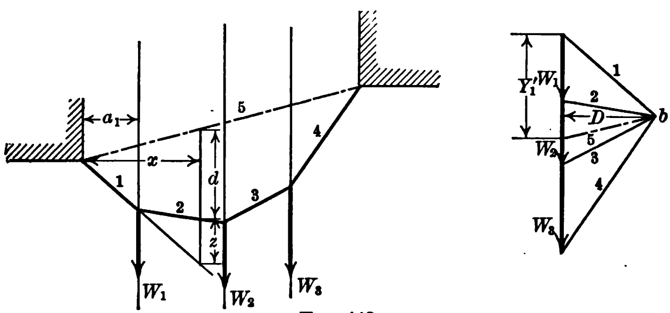


FIG. 143

at a certain point and D the distance of the pole b from the load line (Fig. 143), we have from similar triangles,

$$(d+z) : x :: Y_1' : D,$$

and

$$z : (x - a_1) :: W_1 : D.$$

Eliminating z,

$$d = [xY_1' - (x - a_1)W_1]\frac{1}{D}.$$

Since a_1, Y'_1, and W_1 are constants, and for the given position x is constant, this equation shows that d varies inversely as D. The proof can be given in like manner for any point.

The depth of the string polygon at any point can then be made to have any desired value by a proper choice of

the pole. If with a pole distance D the depth of the string polygon at a certain point is d, and a depth d_1 is desired, choose a new pole distant D_1 from the load line so that $D_1 = \frac{Dd}{d_1}$, and the string polygon constructed by the use of this pole will be the one sought.

Problem 180. Weights of 100, 300, 200 lb. are to be suspended from a cord in lines 3, 5, and 8 ft. respectively from the left support. The right support is distant 10 ft. horizontally and 2 ft. vertically above the left support. Using a scale of 1 in. = 2 ft., construct a string polygon whose depth shall be 4 ft. at the second load. From the diagram scale off the tensions in each part of the cord and the horizontal component of the stress in the cord. Use a force scale of 1 in. = 100 lb.

Problem 181. A cord is suspended from two points on the same level 9 ft. apart. Eight weights of 50 lb. each are to be suspended at equal intervals between the supports. The maximum tension in the cord is to be 300 lb. Draw the shape of the cord. Find the total length of the cord and the horizontal component of the tension.

Problem 182. Show that when equal weights are distributed at equal horizontal intervals along a cord suspended between two supports on the same level, the maximum tension in the cord is greater than half the total load on the cord.

Problem 183. A cord supported at the ends on the same level carries a load uniformly distributed along the horizontal. Draw approximately the shape of the cord, given that the tension of the cord at the lowest point is one half the total load.

SUGGESTION. Divide the horizontal distance up into small equal intervals and assume that the weight in each interval acts at the center of that interval.

With the vertex at the lowest point of the string polygon, construct a parabola to pass through the points of support and compare it with the string polygon.

Problem 184. Make the constructions of the above problem when the points of support are not on the same level.

Problem 185. Show how to obtain the ratios of a set of weights to cause a cord to hang in a given string polygon.

HINT. Work from the string polygon to the force polygon.

90. The Linked Arch. — Analogous to the problem of finding the form taken by a cord carrying loads at intervals is that of finding the form of a set of connected weightless links to sustain a given set of loads in given vertical lines, the links to be in compression instead of tension. Here again if the horizontal component of the

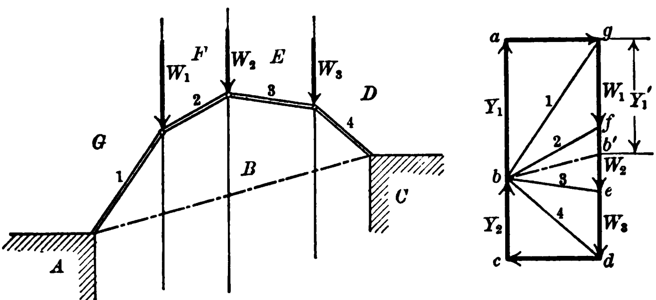


FIG. 144

thrust in any link is given, the form of the linkage can be determined. The construction and proof are analogous to those for the cord carrying weights. (See Fig. 144.) The links themselves lie in an equilibrium polygon the pole of which is b. It will be noted that the pole of the link polygon lies on the opposite side of the load line from the pole of the string polygon, and that all poles of link polygons lie on the same straight line as that on which the poles of the string polygons lie.

91. The Masonry Voussoir Arch. — In an arch made of links, constructed to carry a given set of loads, any variation of the loads, unless they were all changed simultaneously in the same ratio, would be apt to cause the collapse of the arch. In an arch built of dressed stone, called *voussoirs*, owing to the size of the bearing surfaces and the friction between the surfaces, an arch can be built to carry a given set of loads and allow for a given varia-

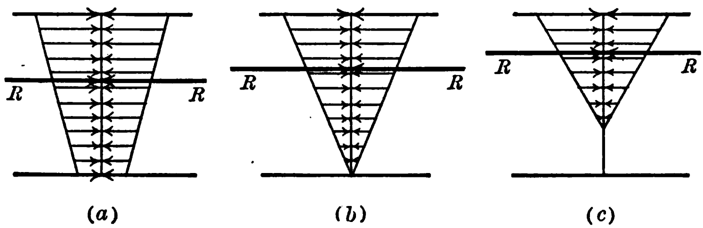


FIG. 145

tion in the loads without endangering the structure. The voussoirs can be made of such a depth as to keep the line of the resultant pressure of two adjacent voussoirs on each other in a certain portion of the joint. The broken line joining the points of the resultant pressures of the adjacent arch stones on each other is called the *line of resistance.* It is usually considered advisable, though it is not necessary for the stability of the arch, to keep the line of resistance within the middle third of the arch. The intensity of the pressure between two arch stones is assumed to vary uniformly as indicated in Fig. 145. In (*a*) the resultant pressure falls within the middle third, in (*b*) at the edge of the middle third, and in (*c*) outside the middle third. In the latter case there is a tendency

for the joint to open. With the line of resistance below the middle third there would be a tendency to open the joint on the upper part of the arch, and also to unduly increase the pressure on the lower part of the joint. Where the line of resistance falls above the middle third the joint would tend to open on the lower side.

It is desired here to present only the elementary principle of the arch. An adequate treatment of arches may be found in I. O. Baker's "Treatise on Masonry" (John Wiley & Sons).

Problem 186. Find the shape of a linkage that would carry the weights of Problem 178 with the same horizontal stress, the links being in compression. Is the link polygon the same as the inverted string polygon of that problem?

Problem 187. Show that when, and only when, the supports are on the same level, the link polygon for a given set of loads and horizontal stress is the inverted string polygon for that set of loads. Show also that in any case the link polygon can be obtained from the string polygon turned through 180° in its plane if in constructing the string polygon the loads and their horizontal distances are reversed in order from right to left.

Problem 188. An arch is to be constructed with its center line in the form of an arc of a circle, the span being 20 ft., and rising 6 ft. in the middle. Considering the load on the arch to be vertical only and at any point proportional to the distance from the center line of the arch to a straight horizontal line 6 ft. above the highest point of the center line, sketch approximately the line of resistance. About how deep would the arch stones need to be to keep the line of resistance within the middle third?

SUGGESTION. Assume the loads concentrated at equal horizontal intervals and construct the link polygon with the required height of 6 ft. It may then be seen how the line of resistance may be changed

so as to more nearly coincide with the given circle by a slight movement of the pole.

Problem 189. Find approximately by graphical methods the form of an arch to carry a load uniformly distributed along the horizontal.

92. Bending Moments. — As an application of the equilibrium polygon for parallel forces a brief discussion is here given of bending moments in a horizontal beam subjected to vertical forces only. Let Fig. 146 represent a horizontal beam acted on by vertical forces. At any point C pass a vertical plane perpendicular to the axis of the beam.

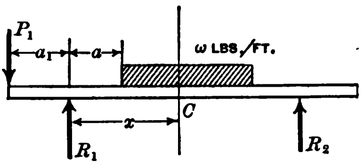


FIG. 146

The sum of the moments about a horizontal line in this plane of all the forces acting on the beam to the left of the plane is called the *bending moment* at that section. Clockwise direction will be counted as positive. Representing the bending moment by M, the value of M at C (Fig. 146) is

$$M = R_1 x - (x + a_1)P_1 - \frac{W(x-a)^2}{2}.$$

Since the sum of the moments of all the forces acting on the beam is zero, the sum of the moments of the forces to the right of the section is the negative of the sum of the moments to the left of the section. Hence, in computing bending moments, if it is more convenient, we may take as the bending moment at the section the sum of the moments of all forces acting on the beam to the right of the section and consider counter-clockwise as positive.

Problem 190. A beam 10 ft. long is supported at the ends and carries loads of 500 lb., 600 lb., 800 lb. at distances of 3 ft., 5 ft., and 8 ft. respectively from the left end. Compute the bending moment at intervals of 1 ft. along the beam, neglecting the weight of the beam.

Problem 191. A beam 12 ft. long, supported at the ends, carries a load of 500 lb. at a point 4 ft. from the left end and a uniformly distributed load of 200 lb. per linear foot over the right half of the beam. The beam weighs 30 lb. per foot of length. Compute the bending moment at intervals of 2 ft. along the beam. Using M as ordinate and distance along the beam as abscissa, sketch a curve representing the bending moment at all sections.

93. Bending Moment Diagram. — Using the bending moment as ordinate and distance along the beam as abscissa, a curve may be plotted which shows the bending moment

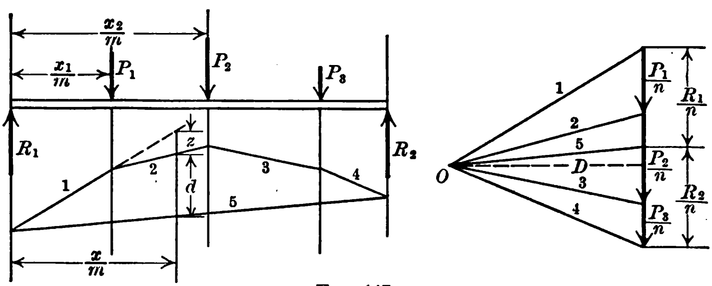


FIG. 147

at each section. This curve is called the bending moment diagram. It will now be shown that for concentrated loads the equilibrium polygon represents to a certain scale the bending moment diagram.

Consider a beam supported at the ends carrying loads P_1, P_2, P_3 lb. at distances of x_1, x_2, x_3 ft. from the left end

respectively. Construct the equilibrium polygon for the forces and reactions, using the following scale:

$$1'' = m \text{ ft. along the beam,}$$
$$1'' = n \text{ lb. on the line of loads.}$$

Let D be the pole distance in inches, and d the depth in inches of the equilibrium polygon at a point distant $\frac{x}{m}$ in. from the left support between the loads P_1 and P_2.

By similar triangles

$$(d+z) : \frac{x}{m} = \frac{R_1}{n} : D,$$

and

$$z : \frac{x - x_1}{m} = \frac{P_1}{n} : D.$$

Eliminating z,

$$d = \frac{xR_1 - (x - x_1)P_1}{mnD}.$$

By definition the bending moment, M, measured in pound-feet at the given point is

$$M = xR_1 - (x - x_1)P_1.$$

$$\therefore d = \frac{M}{mnD},$$

or

$$M = mnD \cdot d.$$

This formula can as easily be shown to hold for any other point. Hence *the depth of the equilibrium polygon in inches multiplied by the number of ft. per inch along the beam, the number of lb. per inch on the load line, and the pole distance in inches gives the value of the bending moment in lb.-ft.*

The depth of the equilibrium polygon then represents the bending moment to the scale

$$1'' = mnD \text{ lb.-ft.}$$

If it is desired to use the pound-inch as the unit of moment, then let $1'' = m$ in. along the beam.

To obtain the bending moment diagram to a given scale, $1'' = k$ lb.-ft., it is only necessary to choose D so that $mnD = k$.

For a distributed load the bending moment curve may be obtained approximately by dividing the beam length into small intervals and considering the load on each interval to act at the center of that interval.

Problem 192. Construct graphically the bending moment for the beam of Problem 190. Let 1 in. = 2 ft. along the beam, 1 in. = 500 lb. on the load line, and choose D so that 1 in. on the moment diagram shall equal 1500 lb.-ft. of moment.

Problem 193. Construct approximately the bending moment diagram for a beam supported at the ends and carrying a uniformly distributed load.

94. Bending Moment for Beams not Supported at the Ends. — (*a*) *Cantilever Beam.* A cantilever beam is shown in Fig. 148. The construction for the bending moment is indicated, the scale factors being the same as in Art. 93.

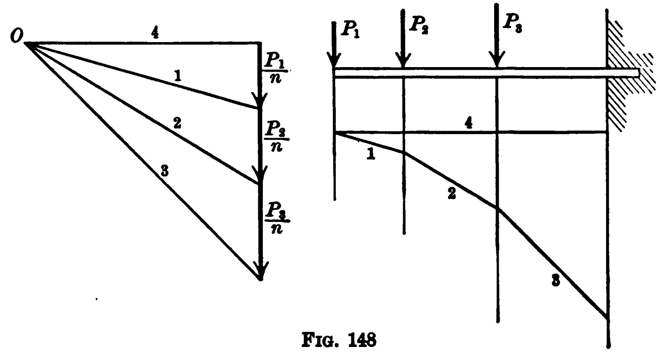


FIG. 148

(*b*) *Overhanging Beam.* In Fig. 149 the construction is indicated for the bending moment of an overhanging beam. The diagram (*i*) is the equilibrium polygon. The diagram (*ii*) is obtained from (*i*) by replacing the closing line of the polygon and the two lines connected

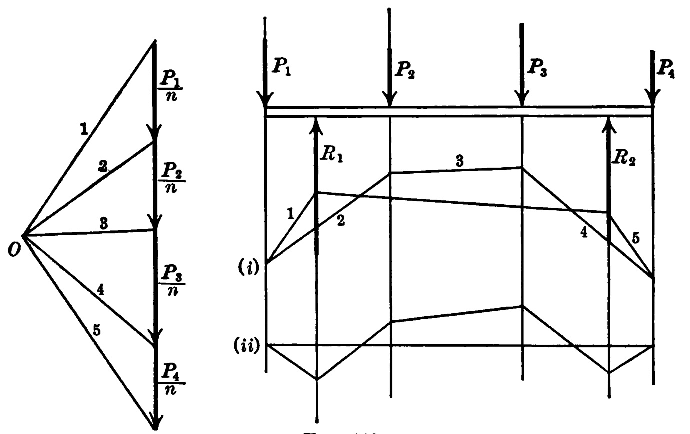


FIG. 149

with it by a horizontal line and drawing a polygon with a depth on each load line and reaction line, the same as that of the equilibrium polygon (*i*).

Problem 194. Prove that the method indicated of drawing the bending moment diagram of a cantilever beam is correct.

Problem 195. Prove that the method indicated of constructing the bending moment diagram of the overhanging beam is correct.

Problem 196. Construct the bending moment diagram of a cantilever beam 12 ft. long carrying loads of 400, 600, and 1000 lb. at distances of 0, 3, and 7 ft. respectively from the free end. Scale the bending moment at 3 ft. and at 6 ft. from the free end and compare with computed values. State the scales used.

Problem 197. A beam 16 ft. long is supported at two points distant respectively 3 ft. from the left end and 4 ft. from the right end. Loads of 300, 500, 1000, and 600 lb. are applied at distances of 0, 6, 10, and 16 ft. respectively from the left end. Construct graphically the bending moment diagram to a convenient scale. Check by computing analytically the bending moment at several points.

95. Tensile and Compressive Stresses in Beams. — Let Fig. 150 represent the portion of the beam of Fig. 146 to the left of the plane section at C. This portion of the beam is in equilibrium under the action of the external forces acting on this portion and the internal forces exerted on it by the portion of the beam to the right of the section. These internal forces can be resolved into horizontal and vertical forces. The vertical component of the internal forces must be equal and opposite to the sum of the external forces to the left of the section. This latter sum is called the *shear* at the section.

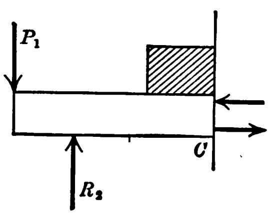


Fig. 150

Since the external forces are all vertical, the sum of the horizontal components of the internal forces must be zero. The horizontal forces therefore form a couple. This couple is called the *internal stress couple.* To produce equilibrium the moment of the internal stress couple must be equal and opposite in sign to the moment about any horizontal line in the section of all the external forces to the left of the section. That is, the internal stress couple at any section of the beam is equal to the bending moment at that section changed in sign.

It can be shown by experiment that when a rod is stretched or compressed within the elastic limit, the amount of elongation or compression is directly proportional to the applied force. Defining *unit stress* in a rod uniformly stretched or compressed, *i.e.* without bending, as the total force divided by the cross-sectional area, and *unit strain* as the amount of elongation or compression divided by the original length, this experimental fact may be expressed by the equation,

$$\frac{\text{unit stress}}{\text{unit strain}} = E$$

where E is constant for a given material.

E is called the *modulus of elasticity*. Experiment shows that it is the same for compression as for tension, for such materials as wood and structural steel.

It is also shown by experiment that when a beam is bent, within the elastic limit, a plane section of the beam before bending remains a plane section after bending. When the beam is bent, the upper fibers of the beam are then compressed (or elongated) and the lower fibers are elongated (or compressed) as in Fig. 151. There will be

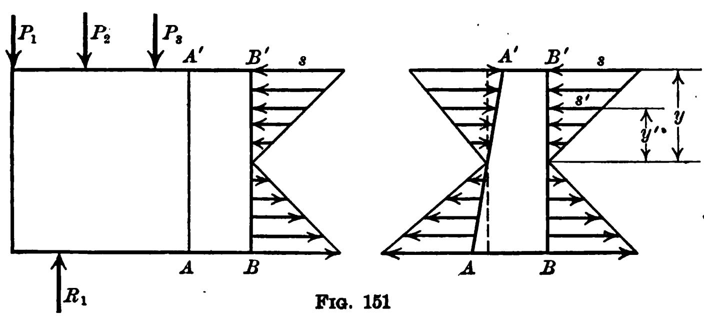


FIG. 151

somewhere between top and bottom a surface, called the *neutral surface*, where there is neither tension nor compression, and the amount of tensile or compressive stress at any point in the section will vary directly as the distance of this point from the neutral surface, since the amount of the elongation or compression of the longitudinal fiber of the beam through the point varies as the distance from the neutral surface.

If s is the unit compressive stress at the upper outside fiber of the beam, distant y from the neutral surface, s' the unit stress at a distance y' from the neutral surface, then

$$s' = \frac{y'}{y} s.$$

s' will be positive or negative according to the sign of y'.

Considering the portion of the beam as in equilibrium under the action of the external and internal forces, if dA is an element of area of the cross section distant y' from the neutral surface, we have from $\Sigma X = 0$,

$$\int s' dA = 0,$$

or

$$\frac{s}{y}\int y' dA = 0.$$

But $\int y' dA$ is equal to the product of the area of the cross section and the ordinate of the center of gravity measured from the origin of ordinates, *i.e.* from the neutral surface. If $\bar{y}$ is this ordinate, then

$$\frac{s}{y}\bar{y}A = 0.$$

Therefore

$$\bar{y} = 0,$$

and the neutral surface passes through the center of gravity of the section.*

Equating moments of external and internal forces about a horizontal gravity axis of the section,

Mom of external forces = −mom of internal forces,

or
$$M = \int y's'dA$$
$$= \frac{s}{y}\int y'^2 dA = \frac{s}{y} I$$

where I is the moment of inertia of the area of the cross section with respect to the horizontal gravity axis of the section.

$$\therefore \qquad s = \frac{My}{I}.$$

This formula gives the stress s at a distance y from the neutral axis of the section. The stress at any other point in the section is obtained by the direct proportion

$$\frac{s'}{s} = \frac{y'}{y}.$$

For a beam of constant cross section y and I are constant and hence s varies directly as M. The maximum tensile or compressive stress will therefore be found where the bending moment has maximum numerical values.

In computing stresses care must be used to have the same units throughout the formula. If stress is reckoned in pounds per square inch, the dimensions of the cross

* In the above it is assumed that the intersection of the vertical section and the neutral surface is a horizontal straight line. If the loads lie in one plane which is a plane of symmetry of the beam, this will be the case. Under other conditions the neutral surface may not be horizontal.

section must be in inches and the bending moment in pounds-inches.

Problem 198. A beam 4 in. broad by 6 in. deep, 12 ft. long, is supported at the ends and carries loads of 400 lb. and 800 lb. at distances of 5 ft. and 9 ft. respectively from the left end. Sketch the bending moment diagram and find the maximum fiber stress.

Ans. 1150 lb.-sq. in. under the 800 lb. load.

Problem 199. A beam 4 in. broad by 6 in. deep, 12 ft. long, is supported at the ends and carries two equal loads at distances of 4 ft. from the ends. What maximum value may the loads have if the maximum allowable fiber stress is 1000 lb. per sq. in.?

Problem 200. Find the ratio of the loads that could be put on a beam 4″ by 6″ when placed with the 4″ face horizontal and when placed with the 6″ face horizontal, the maximum fiber stress to be the same in the two cases.

Problem 201. Compare the loads, one concentrated at the middle, the other uniformly distributed over the length of the beam, to cause the same maximum fiber stress in the beam.

Problem 202. An I-beam of depth 10 in. and weight 25 lb. per linear foot is 16 ft. long and is supported at the ends. What concentrated load can it carry in the middle so as to cause a maximum fiber stress of 16,000 lb. per sq. in.? The moment of inertia of the cross section about a horizontal gravity axis is 122 in.^{4}. Take the weight of the beam into consideration.

96. Cords and Pulleys. — When a cord passes over a pulley, without friction, the tension is transmitted along its length undiminished. A weight W attached to a cord which passes vertically over a pulley is raised by a direct downward pull P on the other end of the rope. If there is no friction, P is equal to W, for uniform motion. In the case of a system of pulleys, as shown in Fig. 152, the

cord may be considered as under the same tension throughout and parallel to itself in passing from one sheave to the other. It is then possible to cut across the cords, just as was done in the case of the bridge truss, Problem 92, where the stress was along the member in each case. Cutting all the cords at C and considering all the forces acting on the sheave B, we get, calling the tension in the cord P,

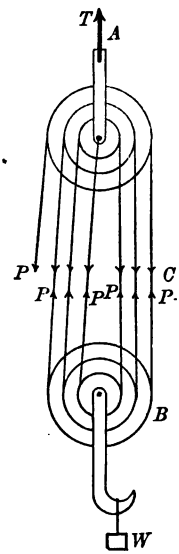


FIG. 152

$$6\,P = W,$$

or the tension in the cord is $W/6$. A consideration of the upper sheaves gives $T = 7\,P = 7/6\,(W)$. The various cases of cords and pulleys that come up in engineering work may be taken up in a similar way, but in any case of cutting cords, it must be remembered that all cords attaching one part to another must be cut and the tension acting along the cords inserted before the principles of equilibrium can be applied. The consideration of the friction between cords and pulleys will be taken up in Chapter XIII.

Problem 203. Suppose the hook of the lower sheave of Fig. 152 attached to a weight of 350 lb., and a man of weight 150 lb. stands on the weight and lifts himself and the weight by pulling on the rope. What force must he exert? If he stands on the ground, what force must he exert to raise a weight of 500 lb.?

97. Equilibrium of a Flexible Cord Carrying a Load Uniformly Distributed along the Horizontal. — Let the cord supported at

A and B (Fig. 153 (a) and (b)) carry a uniform load w per unit length along the horizontal. Consider a small portion of the cord, Δs, with ends at (x, y) and $(x + \Delta x, y + \Delta y)$, the coördinate axes being horizontal and vertical. This

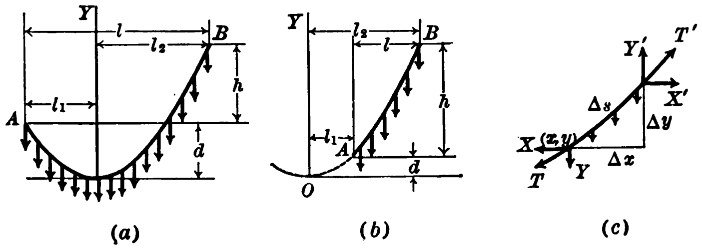


FIG. 153

portion is in equilibrium under the action of the tensions T and T' at its ends and the load $w\Delta x$ carried by this portion of the cord (Fig. 153 (c)).

Resolving T and T' into horizontal and vertical components and applying the conditions for equilibrium, we get

$$X' - X = 0, \tag{1}$$

$$Y' - Y = w\Delta x. \tag{2}$$

But $\qquad Y' - Y =$ increment of $Y = \Delta Y$.

Equation (1) shows that the horizontal component of the tension is the same throughout the cord, and equation (2) may be written

$$\frac{dY}{dx} = w,$$

and hence, $$Y = wx + c. \tag{3}$$

Now X and Y are respectively equal to $T \cos\theta$ and $T \sin\theta$, where θ is the inclination of the tangent line to the

curve at (x, y). Equations (1) and (3) may then be written

$$T \cos \theta = X,$$
$$T \sin \theta = wx + c.$$

By division we get, since $\tan \theta = \frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{wx + c}{X}. \qquad (4)$$

Since c and w are constants, a value of x can be found for a point on the curve, not necessarily on the cord between A and B, for which $\frac{dy}{dx} = 0$. If the origin be moved to this point on the curve by a translation of axes, equation (4) will take the form

$$\frac{dy}{dx} = \frac{w}{X} x. \qquad (5)$$

This means that the origin is taken at the lowest point of the curve as in Fig. 153 (*a*) and (*b*). In (*a*) the origin is on the cord, and in (*b*) it is not.

Integrating equation (5),

$$y = \frac{w}{2X} x^2 + c'.$$

But $y = 0$ when $x = 0$. Therefore $c' = 0$ and the equation of the curve becomes

$$y = \frac{w}{2X} x^2.$$

Hence the cord hangs in a portion of a parabola with vertical axis.

98. The Horizontal Component of the Tension in Terms of the Length of Span and the Deflection.

The horizontal distance between the supports is called the *span*, l.

The vertical distance of the vertex of the parabola below the lower point of support is called the *deflection*, d.

In Fig. 153 (a) and (b) the coördinates of A are $(-l_1, d)$ and (l_1, d) respectively, and the coördinates of B are $(l_2, d+h)$. The coördinates of A and B satisfy the equation of the curve. Hence in case (a)

$$d = \frac{w}{2X} l_1^2, \text{ and } d + h = \frac{w}{2X} l_2^2.$$

Therefore, $$l_1 = \sqrt{\frac{2Xd}{w}}, \quad l_2 = \sqrt{\frac{2X(d+h)}{w}}.$$

Therefore $$l = l_1 + l_2 = \sqrt{\frac{2X}{w}}(\sqrt{d} + \sqrt{d+h}),$$

and $$X = \frac{wl^2}{2(\sqrt{d+h} + \sqrt{d})^2}.$$

If $h = 0$, the supports are on the same level and

$$X = \frac{wl^2}{8d}.$$

In case (b), $$l = l_2 - l_1 = \sqrt{\frac{2X}{w}(d+h)} - \sqrt{\frac{2Xd}{w}},$$

and $$X = \frac{wl^2}{2(\sqrt{d+h} - \sqrt{d})^2}.$$

99. Length of Cord. — The length of the curve from the origin to any point (x, y) on the curve is given by

$$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx.$$

Letting $\dfrac{X}{w} = a$, the equation of the parabola becomes $y = \dfrac{x^2}{2a}$, from which $\dfrac{dy}{dx} = \dfrac{x}{a}$, and hence

$$s = \frac{1}{a}\int_0^x \sqrt{a^2 + x^2} \cdot dx$$

$$= \frac{1}{2a}\Big[x\sqrt{a^2 + x^2} + a^2 \log_e (x + \sqrt{a^2 + x^2})\Big]_0^x.$$

$$\therefore s_1 = OA = \frac{1}{2a}\left[l_1\sqrt{a^2 + l_1^2} + a^2 \log_e \frac{l_1 + \sqrt{a^2 + l_1^2}}{a}\right],$$

$$s_2 = OB = \frac{1}{2a}\left[l_2\sqrt{a^2 + l_2^2} + a^2 \log_e \frac{l_2 + \sqrt{a^2 + l_2^2}}{a}\right]$$

The total length of the curve is

in case (*a*) $S = s_1 + s_2$, and

in case (*b*) $S = s_2 - s_1$.

Another formula for the length of the cord may be obtained by expanding $\sqrt{1 + \dfrac{x^2}{a^2}}$ by the binomial theorem and integrating the series term by term when a is less than each of the quantities l_1 and l_2.

Thus, $$s_1 = \int_0^{l_1}\left(1 + \frac{x^2}{a^2}\right)^{\frac{1}{2}} dx$$

$$= \int_0^{l_1}\left(1 + \frac{1}{2}\frac{x^2}{a^2} - \frac{1}{8}\frac{x^4}{a^4} + \frac{1}{16}\frac{x^6}{a^6} - \frac{5}{128}\frac{x^8}{a^8} + \cdots\right)dx$$

$$= l_1 + \frac{1}{6}\frac{l_1^3}{a^2} - \frac{1}{40}\frac{l_1^5}{a^4} + \frac{1}{112}\frac{l_1^7}{a^6} - \frac{5}{1152}\frac{l_1^9}{a^8} + \cdots.$$

In terms of the deflection, d, this becomes, since $a = \dfrac{l_1^2}{2d}$,

$$s_1 = l_1 + \frac{2}{3}\frac{d^2}{l_1} - \frac{2}{5}\frac{d^4}{l_1^3} + \frac{4}{7}\frac{d^6}{l_1^5} - \frac{10}{9}\frac{d^8}{l_1^7} + \cdots.$$

In like manner, if $d_1 = d + h$,

$$s_2 = l_2 + \frac{2}{3}\frac{d_1^2}{l_2} - \frac{2}{5}\frac{d_1^4}{l_2^3} + \frac{4}{7}\frac{d_1^6}{l_2^5} - \frac{10}{9}\frac{d_1^8}{l_2^7} + \cdots.$$

If the supports are on the same level, $l_1 = l_2 = \frac{l}{2}$ and the expressions for the total length S become respectively

$$S = \frac{1}{a}\left[\frac{l}{4}\sqrt{l^2 + 4a^2} + a^2 \log_e \frac{l + \sqrt{l^2 + 4a^2}}{2a}\right],$$

$$S = l + \frac{1}{3}\frac{l^3}{2^3 \cdot a^2} - \frac{1}{20}\frac{l^5}{2^5 \cdot a^4} + \frac{1}{56}\frac{l^7}{2^7 \cdot a^6} - \frac{5}{576}\frac{l^9}{2^9 \cdot a^8} + \cdots,$$

and $S = l + \dfrac{2^3 \cdot d^2}{3\,l} - \dfrac{2^5 \cdot d^4}{5\,l^3} + \dfrac{2 \cdot 2^7 \cdot d^6}{7\,l^5} - \dfrac{5 \cdot 2^9 \cdot d}{9\,l^7} + \cdots.$

For cords for which the deflection is small compared to the length of span the series converge rapidly and three or four terms are sufficient for computing the length. For large values of d the series should not be used.

Problem 204. A suspension bridge as shown in Fig. 154 has a span of 1200 ft. and the cable a maximum deflection at the center $d = 120$ ft. The weight of the floor is 2 tons per linear foot. Find

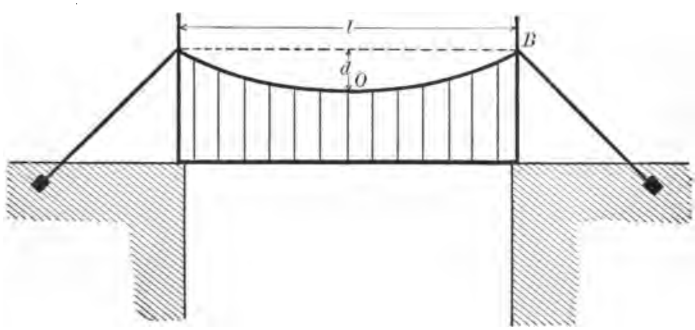


FIG. 154

the equation of the cable and the tension at O and at B. If the safe strength of cable is 75,000 lb. per sq. in., find the area of wire section of cable necessary to support the floor.

Problem 205. Find the length of the cable in the preceding problem.

Problem 206. A cable is to be suspended from two points distant 100 ft. apart horizontally and 20 ft. vertically. It is to carry a load of 500 lb. per horizontal foot, and the lowest point of the curve is to be 25 ft. below the lower point of support. Find the position of the lowest point of the cord, the value of the horizontal component of the tension, and the maximum tension.

Problem 207. Find the length of the cable of the preceding problem.

Problem 208. A cable 105 ft. long, carrying a uniform horizontal load, is stretched between two points on the same level distant 100 ft. apart. Find the deflection.

SUGGESTION. Use the first three terms of the series which expresses S in terms of l and d. See how large the fourth term is with the computed value of d.

Problem 209. A uniform wire weighing $\frac{1}{2}$ lb. per foot of length is supported between two points 200 ft. apart on the same level and the maximum deflection is $2\frac{1}{2}$ ft. Find the horizontal tension, the maximum tension, and the length of the wire.

SUGGESTION. Since the deflection is here relatively small, the load is very nearly uniformly distributed along the horizontal. The curve may then be regarded as approximately a parabola.

100. Equilibrium of a Flexible Cord Carrying a Load Uniformly Distributed along the Cord.—Using a notation like that of Art. 97, the same method leads to an equation like (5) of that article, with x replaced by s; namely,

$$\frac{dy}{dx} = \frac{w}{X}s,$$

in which the origin is at a point of the curve where the slope is zero and s is the length of the curve from the origin to the point (x, y).

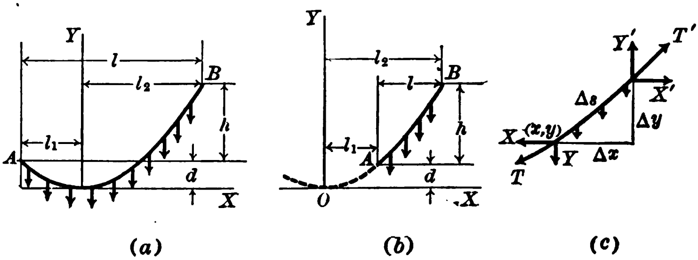


FIG. 155

Replacing dy by $\sqrt{ds^2 - dx^2}$ and writing $\dfrac{X}{w} = a$, this becomes

$$\frac{ds^2 - dx^2}{dx^2} = \frac{s^2}{a^2},$$

or

$$ds = \frac{1}{a}\sqrt{a^2 + s^2} \cdot dx.$$

$$\therefore \quad \frac{ds}{\sqrt{a^2 + s^2}} = \frac{dx}{a}.$$

Integrating, $\quad \log_e (s + \sqrt{s^2 + a^2}) = \dfrac{x}{a} + c.$

$$s = 0 \text{ when } x = 0 \text{ ; } \therefore c = \log a.$$

$$\therefore \quad \log_e \left(\frac{s + \sqrt{a^2 + s^2}}{a}\right) = \frac{x}{a}.$$

$$\therefore \quad \frac{s + \sqrt{a^2 + s^2}}{a} = e^{\frac{x}{a}}.$$

Invert, $$\frac{a}{s + \sqrt{a^2 + s^2}} = e^{-\frac{x}{a}}.$$

Rationalizing the denominator,

$$\frac{s - \sqrt{a^2 + s^2}}{a} = -e^{-\frac{x}{a}}.$$

Add to $$\frac{s + \sqrt{a^2 + s^2}}{a} = e^{\frac{x}{a}}.$$

Then $$s = \frac{a}{2}(e^{\frac{x}{a}} - e^{-\frac{x}{a}}). \qquad (1)$$

It was found above that $\frac{dy}{dx} = \frac{s}{a}$.

$\therefore$ $$dy = \tfrac{1}{2}(e^{\frac{x}{a}} - e^{-\frac{x}{a}})dx$$

from which $$y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}}) + c'.$$

But $x = 0$ when $y = 0$; $\therefore c' = -a$.

$$\therefore y + a = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}}), \qquad (2)$$

which is therefore the equation of the curve, the origin being at the lowest point of the curve (Fig. 155 (*a*) and (*b*)).

This curve is called the *catenary*.

The length of the curve follows at once from equation (1); namely,

$$OA = s_1 = \frac{a}{2}(e^{\frac{h_1}{a}} - e^{-\frac{h_1}{a}})$$

and $$OB = s_2 = \frac{a}{2}(e^{\frac{h_2}{a}} - e^{-\frac{h_2}{a}}), \qquad \text{(Fig. 155)}$$

and $S = s_2 \pm s_1$ according as the cord hangs as in (*a*) or (*b*) of Fig. 155.

If the points of support are on the same level, $l_1 = l_2 = \frac{l}{2}$, and the total length of cord is

$$S = a\left(e^{\frac{l}{2a}} - e^{-\frac{l}{2a}}\right).$$

The values of y and s may be expressed in a series by making use of the expansion

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{x^5}{\underline{|5}} + \cdots.$$

It is left for the student to show that

$$y = \frac{1}{2}\frac{x^2}{a} + \frac{1}{\underline{|4}}\frac{x^4}{a^3} + \frac{1}{\underline{|6}}\frac{x^6}{a^5} + \cdots.$$

$$s_1 = l_1 + \frac{1}{\underline{|3}}\frac{l_1^3}{a^2} + \frac{1}{\underline{|5}}\frac{l_1^5}{a^4} + \cdots.$$

101. Representation by Means of Hyperbolic Functions. — From the definitions

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}),$$
$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$

it follows at once that the values of y and s, the arc from O to the point (x, y), are given by the formulæ

$$y + a = a\cosh\frac{x}{a},$$

$$s = a\sinh\frac{x}{a}, \qquad \text{where } a = \frac{X}{w}.$$

A table of hyperbolic sines and cosines is given in the appendix and should be used in the solution of the following problems.

Problem 210. A flexible wire weighing $\frac{1}{2}$ lb. per foot is supported by two posts 200 ft. apart. The horizontal pull on the wire is 500 lb. Find the deflection at the center and the length of the wire.

Problem 211. What pull will be necessary in Problem 210 so that the greatest deflection will not be greater than 6 in.? What is the length of the wire for this case?

Problem 212. Find the tension in the wire of Problem 210 at the supports.

Problem 213. A wire weighing $\frac{1}{4}$ lb. per foot is suspended between two points A and B where B is 20 ft. higher than A and distant 120 ft. horizontally. The horizontal component of the tension of the wire is 50 lb. Find the position of the lowest point of the curve, the deflection, the length of the wire, and the maximum tension. Sketch the curve in which the wire hangs.

SUGGESTION. Assuming that the wire hangs as in (*a*), Fig. 155, we may write

$$d + 20 = 200 \cosh \frac{l_2}{200}, \qquad \text{since } a = \frac{X}{w} = 200,$$

$$d = 200 \cosh \frac{l_1}{200}.$$

Subtracting, $$20 = 200 \left(\cosh \frac{l_2}{200} - \cosh \frac{l_1}{200}\right)$$

$$= 400 \sinh \frac{l_2 + l_1}{400} \sinh \frac{l_2 - l_1}{400},$$

since $$\cosh x - \cosh y = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2}.$$

But $$l_1 + l_2 = 120.$$

Therefore $$20 = 400 \sinh .3 \cdot \sinh \frac{l_2 - l_1}{400},$$

from which the value of $l_2 - l_1$ may be found from the table of hyperbolic sines. Combining this value with $l_2 + l_1 = 120$, the values of l_1 and l_2 are determined. The values of d, S, and the maximum tension may then be determined.

Problem 214. Solve the preceding problem if B is 40 ft. higher than A, all other conditions remaining unchanged.

Problem 215. A wire is suspended between two points on the same level 240 ft. apart. The deflection at the center is 60 ft. Find the value of a, or $\frac{X}{w}$, on the supposition that the load is (1) uniformly distributed along the horizontal, (2) uniformly distributed along the wire. Compute the values of y for $x = 40$ and for $x = 80$ ft. for both cases and sketch the parabola and the catenary.

Suggestion. In case (1) a is obtained by substituting the coördinates of the point of support in the equation of the parabola. In case (2) the value of a found for the parabola may be used as a trial value and the correct value of a found by modifying the trial value until a value is found to satisfy the equation

$$y + a = a \cosh \frac{x}{a}, \qquad \text{where } y = 60 \text{ and } x = 120.$$

That is, a must satisfy

$$\frac{60}{a} + 1 = \cosh \frac{120}{a}.$$

Problem 216. Solve the preceding problem if the deflection in the middle is only 6 ft.

CHAPTER IX

MOTION IN A STRAIGHT LINE (RECTILINEAR MOTION)

102. Velocity. — When a particle moves along a straight line passing over equal spaces in equal times, it is said to have *uniform motion.* In this case the ratio of the space passed over to the time taken to pass over that space is called the *velocity* of the particle.

If the motion is along a straight line but is not uniform, the ratio of the space passed over in any time to the time is called the average velocity of the particle for that time, or space. Thus, if the particle moves from P to P_1, a distance Δs, in the time Δt, then $\frac{\Delta s}{\Delta t}$ = average velocity of the particle between P and P_1.

The limiting value of $\frac{\Delta s}{\Delta t}$ as Δt approaches the limit zero is defined to be the velocity of the particle at P.

Thus, $$v = \frac{ds}{dt};$$

that is, *the velocity is the first derivative of the distance with respect to time.*

The unit of velocity is the unit of space per unit of time, as *ft. per sec.*, *mi. per hr.*, etc. Speed is sometimes used instead of velocity, especially in speaking of the motion of machines or parts of machines. Speed, however, involves only the rate of motion without reference to its direction, while velocity involves both rate of motion and the direction in which the motion takes place.

Problem 217. The space passed over by a particle moving in a straight line is given by the formula $s = 16\,t^2$, where s is the distance in feet passed over in the time t in seconds. Find the velocity when $t = 0$, when $t = 2$, and at the end of t sec. What is the velocity of the particle when it has moved 40 ft.?

Problem 218. A particle moves back and forth along a straight line, the abscissa of the particle being given by $x = k \cos(ct)$. Find the location of the particle and its velocity when $t = 0, \frac{\pi}{2c}, \frac{\pi}{c}, \frac{3\pi}{c}$. Erect ordinates to represent the velocity for several positions of the moving particle and represent the velocity by a curve.

103. Acceleration for Straight-line Motion. — When the velocity of a particle moving in a straight line increases by equal amounts in equal times, the motion is said to be *uniformly accelerated*, and the gain in velocity per unit of time is called the *acceleration* of the particle.

If, for straight-line motion, the velocity changes from v at the time t to $v + \Delta v$ at the time $t + \Delta t$, then $\frac{\Delta v}{\Delta t}$ is called the *average acceleration* of the particle for the time Δt.

The limiting value of $\frac{\Delta v}{\Delta t}$, *i.e.* $\frac{dv}{dt}$, is defined to be the *acceleration of the particle at the time t.*

Thus, if a represents acceleration,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Another form for a is frequently used. Since $a = \frac{dv}{dt}$, and $v = \frac{ds}{dt}$, there results

$$vdv = ads$$

by eliminating dt.

The unit of acceleration is the unit of velocity per unit of time, as *ft. per sec. per sec.*, *mi. per hr. per hr.*, *yd. per min. per sec.*, etc.

Problem 219. Find the acceleration of the particle of Problem 217 at the times specified. Derive an equation which expresses the velocity in terms of the space, and from this equation obtain a formula for the acceleration in terms of the space.

Problem 220. Using t for abscissa and s, v, a, respectively, as ordinates, plot the curves which represent the space, velocity, and acceleration respectively in terms of the time in Problem 217. Show how the ordinate to the velocity curve is related to the slope, or gradient, of the space curve at any value of t. Likewise for the acceleration and velocity curves.

Problem 221. Find the acceleration of the moving particle in Problem 218. Show that the acceleration is proportional to x. Plot curves showing x, v, and a in terms of t. Also a curve showing a in terms of x.

Problem 222. Show that if a curve is plotted, using values of time as abscissas and values of velocity as ordinates, the area under the curve between two values of the time is equal to the space passed over by the particle in that interval of time.

104. Constant Acceleration. — When the acceleration is constant, we have the relation $dv = a_c dt$, a_c representing the constant value of a, and therefore

$$\int_{v_0}^{v} dv = a_c \int_0^t dt,$$

$$v = a_c t + v_0;$$

and since

$$v = \frac{ds}{dt},$$

$$\int_0^s ds = a_c \int_0^t t\,dt + v_0 \int_0^t dt,$$

or

$$s = \tfrac{1}{2} a_c t^2 + v_0 t.$$

In a similar way the relation

$$vdv = a_c ds$$

gives

$$\int_0^v vdv = a_c \int_0^s ds;$$

therefore

$$\frac{v^2}{2} - \frac{v_0^2}{2} = a_c s,$$

or

$$s = \frac{v^2 - v_0^2}{2\,a_c}.$$

These equations of motion give the velocity in terms of time, the distance in terms of time, and the distance in terms of velocity.

105. Freely Falling Bodies. — Bodies falling toward the earth near its surface have a constant acceleration. It is usually represented by g and equals approximately 32.2 ft. per second per second. The value of g varies slightly with the height above the sea level and the latitude, but for the purposes of engineering it may usually be taken as 32.2. The equations of motion for such bodies are, then,

$$v = gt + v_0,$$
$$s = \tfrac{1}{2}\,gt^2 + v_0 t,$$
$$s = \frac{v^2 - v_0^2}{2\,g}.$$

If the body falls from rest, $v_0 = 0$, and the equations of motion become

$$v = gt,$$
$$s = \tfrac{1}{2}\,gt^2,$$
$$s = \frac{v^2}{2\,g}.$$

This latter is often written $v^2 = 2\,gh$, where $h = s$.

106. Body Projected Vertically Upward. — When a body is projected vertically upward from the earth, the acceleration is constant and equals $-g$. If the velocity of projection is v_0, the equations of motion are

$$v = -gt + v_0t,$$
$$s = -\tfrac{1}{2}gt^2 + v_0t,$$
$$s = \frac{v^2 - v_0^2}{-2g}.$$

Problem 223. A body is projected vertically downward with an initial velocity of 30 ft. per second from a height of 100 ft. Find the time of descent and the velocity with which it strikes the ground.

Problem 224. A body falls from rest and reaches the ground in 6 sec. From what height does it fall, and with what velocity does it strike the ground?

Problem 225. A body is projected vertically upward and rises to the height of 200 ft. Find the velocity of projection v_0 and the time of ascent. Also find the time of descent and the velocity with which the body strikes the ground.

Problem 226. A stone is dropped into a well, and after 2 sec. the sound of the splash is heard. Find the distance to the surface of the water, the velocity of sound being 1127 ft. per second.

Problem 227. A man descending in an elevator whose velocity is 10 ft. per second drops a ball from a height above the elevator floor of 6 ft. How far will the elevator descend before the ball strikes the floor of the elevator?

Problem 228. In the preceding problem, suppose the elevator going up with the same velocity, find the distance the elevator goes before the ball strikes the floor of the elevator.

107. Newton's Laws of Motion. — Three fundamental laws may be laid down which embody all the principles in accordance with which motion takes place. These are

the result of observation and experiment and are known as *Newton's Laws of Motion.*

First Law. Every body remains in a state of rest or of uniform motion in a straight line unless acted upon by some unbalanced force.

Second Law. When a body is acted upon by an unbalanced force, motion takes place along the line of action of the force, and the acceleration is proportional to the force applied and inversely proportional to the mass of the body.

Third Law. To every action of a force there is always an equal and opposite reaction.

The second law states that in case the system of forces acting on the body is unbalanced, the motion is accelerated. Motion takes place in the direction of the resultant force with an acceleration proportional to the force. It also implies that each force of the system produces or tends to produce an acceleration in its own direction proportional to the force. That is to say, each force produces its own effect, regardless of the action of the other forces.

As a result of this latter fact, if a body is acted upon by a force P and the earth's attraction G, we have

$$P : G = a : g,$$

where G is the weight of the body, g the acceleration of gravity, and a the acceleration due to the force P. From this it follows that

$$P = \frac{G}{g} \cdot a = M \cdot a.$$

The quantity $\frac{G}{g}$, in which G is the weight of the body in pounds, and g is in *ft./sec.*^{2}, is called the ***mass*** of the body in "*Engineer's Units.*"

If $g = 32.2$, the Engineer's Unit of mass is equivalent to 32.2 pounds.

108. Motion on an Inclined Plane. — A body (See Fig. 156), of weight G, moves down an inclined plane, without friction, under the action of a force $G \sin\theta$. The acceleration down the plane equals the accelerating force divided by the mass (Art. 107) $= \dfrac{G\sin\theta}{\frac{G}{g}} = g\sin\theta$. The acceleration is

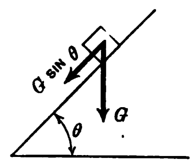


FIG. 156

constant. The equations of motion for such a case, then, are (Art. 104)

$$v = (g\sin\theta)t + v_0,$$
$$s = \frac{1}{2}g(\sin\theta)t^2 + v_0t,$$
$$s = \frac{v^2 - v_0^2}{2g\sin\theta}.$$

If the body starts from rest down the plane, $v_0 = 0$. If it be projected up the plane with an initial velocity v_0, the acceleration equals $-g\sin\theta$.

Problem 229. A body is projected up an inclined plane which makes an angle of 60° with the horizontal with an initial velocity of 12 ft. per second. Neglecting friction of the plane, how far up the plane will the body go? Find the time of going up and of coming down.

Problem 230. A body, weight 20 lb., is projected down the plane given in the preceding problem with a velocity of 20 ft. per second. How far will it go during the third second?

Problem 231. Suppose the body in the preceding problem meets a constant force of friction $F = 10$ lb. What will be the acceleration down the plane? How far will it go during the second second?

Problem 232. A boy who has coasted down hill on a sled has a velocity of 10 mi. per hour when he reaches the foot of the hill. He now goes on a horizontal, meeting a constant resistance of 25 lb. If the combined weight of the boy and sled is 75 lb., how far will he go before coming to rest?

Problem 233. Suppose that in the preceding problem the boy weighs 65 lb. and the sled 10 lb., and that the boy can exert a force of 20 lb. horizontally to keep him on the sled. Will the boy remain on the sled when the latter stops, or will he be thrown forward?

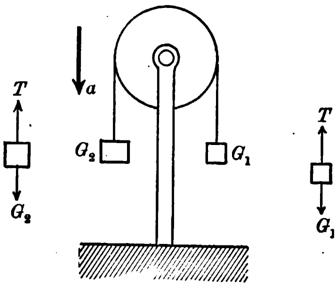


Fig. 157

Problem 234. Two weights, $G_1 = 5$ lb. and $G_2 = 10$ lb. (Fig. 157), attached to an inextensible cord which runs over a pulley, are acted upon by gravity; no friction; motion takes place. Find the tension in the cord, and the acceleration. Consider G_2 and G_1 separately with the forces acting upon them, and call the tension in the cord T. Then apply the principle "*accelerating force equals mass times acceleration.*"

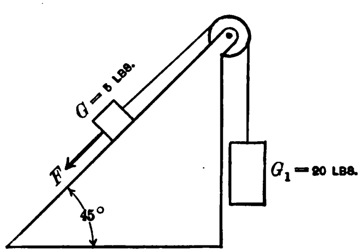


Fig. 158

Problem 235. A body whose weight $G = 5$ lb. is being drawn up an inclined plane as show in Fig. 158 by the action of the weight $G_1 = 20$ lb. Suppose the resistance, F, offered by the plane is 10 lb., and that G starts from rest. How far up the plane will G go in 6 sec.?

Problem 236. An elevator (Fig. 159), whose weight is 2000 lb., is descending with a velocity, at one instant, of 2 ft. per second, and one second later it has a velocity of 18.1 ft. per second. Find the

tension T in the cable that supports the elevator, assuming uniform retardation.

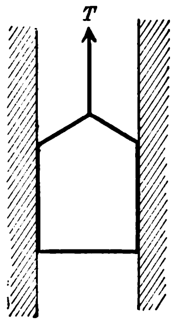


FIG. 159

Problem 237. Suppose the elevator in preceding problem going up with the same acceleration. Find the tension in the cable.

Problem 238. A man can just lift 200 lb. when standing on the ground. How much could he lift when in the moving elevator of the preceding problems, (*a*) when the elevator was ascending? (*b*) when descending?

Problem 239. Two weights, G and G', are connected by an inextensible flexible cord that passes over a frictionless pulley, as shown in Fig. 160. $G = 20$ lb., $G' = 100$ lb., and there is no friction on the plane. Find the tension in the cord and the acceleration of the two bodies.

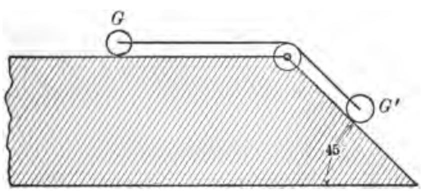


FIG. 160

Problem 240. A 30-ton car is moving with a velocity of 30 mi. per hour on a level track. The brakes refused to work. How far will the car go after the power is turned off before coming to rest, if the friction is .01 of the weight of the car?

109. Variable Acceleration. — It has already been shown that $v = \frac{ds}{dt}$, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$, and $vdv = ads$. These relations hold true no matter whether the acceleration is constant or variable. If the acceleration is *constant*, the equations of motion are those that have already been worked out (Art. 104), and by simple substitution in these equations it is possible to find the velocity in terms of the time, the

distance in terms of time, and the distance in terms of velocity.

If the acceleration is variable, it is necessary to work out the equations of motion for each case. This may be done, when it is known how a varies, by means of either of the equations,

$$a = \frac{d^2s}{dt^2},$$

or

$$vdv = ads.$$

The latter equation will usually give the beginner less difficulty.

110. Harmonic Motion. — Let it be supposed that a body is moved by an attractive force which varies as the distance. That is, the attractive force is proportional to the distance. Then the acceleration is also proportional to the distance.

Let the acceleration $= -ks.$

Then $$vdv = -ksds,$$

and $$\int_{v_0}^{v} vdv = -k\int_0^s sds;$$

therefore $$v^2 - v_0^2 = -ks^2,$$

where v_0 is the initial velocity when s equals zero and k is the factor of proportionality, determinable in any special case. This equation gives the relation between the velocity and distance. Since $v = \sqrt{v_0^2 - ks^2}$, it is evident that $v = 0$ when $\sqrt{k} \cdot s = v_0$. This means that the body comes to rest when s has reached a certain value, viz. $\frac{v_0}{\sqrt{k}}$. From the original assumption, $a = -ks$, it is

seen that the acceleration is greatest when s is greatest, that is, when $s = \frac{v_0}{\sqrt{k}}$; and is least when s is least, that is, when $s = 0$.

To get the relation between distance and time, the equation $v = \sqrt{v_0^2 - ks^2}$ may be put in the form

$$\frac{ds}{\sqrt{v_0^2 - ks^2}} = dt,$$

from which
$$\frac{1}{\sqrt{k}} \sin^{-1} \frac{\sqrt{k} \cdot s}{v_0} = t,$$

or
$$\frac{v_0}{\sqrt{k}} \sin \sqrt{k}t = s.$$

This relation between the distance and time shows that as t increases s changes in value from $\frac{v_0}{\sqrt{k}}$ to $\frac{-v_0}{\sqrt{k}}$, assuming all values between these limits, but never exceeding them, since $\sin \sqrt{k}t$ can never be greater than $+1$ or less than -1. The motion is, therefore, vibratory or periodic, and is known as *harmonic motion.* The complete period in this case is $\frac{2\pi}{\sqrt{k}}$.

The relation between velocity and time may be found for this case by differentiating the last equation with respect to time. Then,

$$v = v_0 \cos \sqrt{k}t.$$

This shows that v_0 is the greatest value of v.

This motion is usually illustrated by imagining a ball attached to two pins by means of two rubber bands or

springs, since the force exerted by either of these is proportional to the elongation. (See Fig. 161.) Assuming that there is no friction and that the ball is displaced to a position B by stretching one of the rubber bands, when released it continues to move backward and forward with harmonic motion.

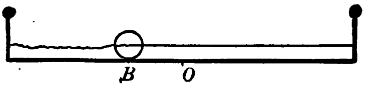


FIG. 161

Problem 241. Suppose the ball in Fig. 161, held by two helical springs, to have a weight of 10 lb. and that it is displaced 1 in. from O. The two springs are free from load when the body is at O. The springs are just alike, and each requires a force of 10 lb. to compress or elongate it 1 in. Find the time of vibration of the body and its velocity and position after $\frac{1}{2}$ sec. from the time when it is released.

It has been found by experiment that the force necessary to compress or elongate a helical spring is proportional to the compression or elongation.

111. Motion with Repulsive Force Acting. — Suppose the force to be one of repulsion and to vary as the distance; then $a = ks$, and $vdv = ksds$, so that

$$v = \sqrt{v_0^2 + ks^2},$$

$$s = \frac{v_0}{2\sqrt{k}}\left(e^{\sqrt{k}\cdot t} - e^{-\sqrt{k}\cdot t}\right) = \frac{v_0}{\sqrt{k}} \sinh \sqrt{k}t.$$

These equations show that as t increases s also increases and the body moves farther and farther away from the center of force. The motion is not oscillatory.

112. Motion where Resistance Varies as Distance. — If a body whose weight is 644 lb. falls freely from rest through 60 ft. and strikes a resisting medium (a shaft where fric-

tion on the sides equals $2F = 10$ times the distance; see Fig. 162), since accelerating force equals mass times acceleration,

$$a = \frac{G - 2F}{M} = \frac{G - 10s}{\frac{G}{g}} = g - \frac{s}{2}.$$

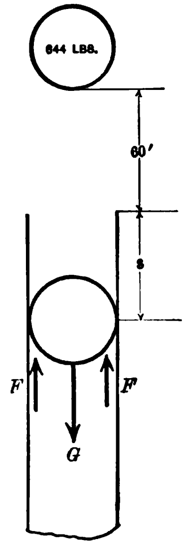


FIG. 162

It is required to find (*a*) the distance the body goes down the shaft before coming to rest; (*b*) the distance at which the velocity is a maximum; (*c*) the total time of fall; (*d*) the velocity at a distance of 10 ft. down the shaft.

After striking the shaft the relation between velocity and distance is as follows:

$$\int_{v_0}^{v} v dv = \int_0^s \left(g - \frac{s}{2}\right) ds.$$

The remainder of the problem is left as an exercise for the student.

Problem 242. A ball whose weight is 32.2 lb. falls freely from rest through a distance of 10 ft. and strikes a 400-lb. spring (Fig. 163). Find the compression in the spring. It is to be understood that a 400-lb. spring is such a spring that 400 lb. resting upon it compresses it one inch, and 4800 lb. resting on it compresses it one foot, if such compression is possible. After the ball strikes the spring it is acted upon by the attraction of the earth and the resistance of the spring. The acceleration a is then $\frac{G - 4800s}{M}$, where s is measured in feet. The relation between velocity and distance is then obtained from the relation,

$$\int_{v_0}^{v} v dv = \int_0^s (g - 4800 s) ds.$$

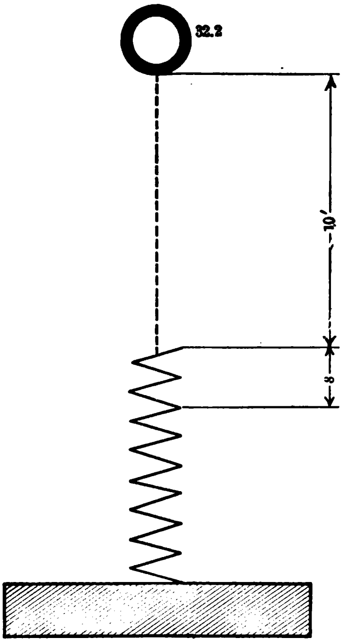


FIG. 163

Problem 243. A 20-ton freight car (Fig. 164), moving with a velocity of 4 mi. per hour, strikes a bumping post. The 60,000-lb. spring of the draft rigging of the car is compressed. Find the compression *s*. Assume that the bumping post absorbs none of the shock.

Problem 244. Suppose the car in the preceding problem to be moving with a velocity of 4 mi. per hour, what should be the strength of the spring in the draft rigging so that the compression cannot exceed 2 in.?

Problem 245. After the spring in Problem 242 has been compressed so that the ball comes to rest, it begins to regain its original form. Find the time required to do this and the velocity with which the ball is thrown from the spring.

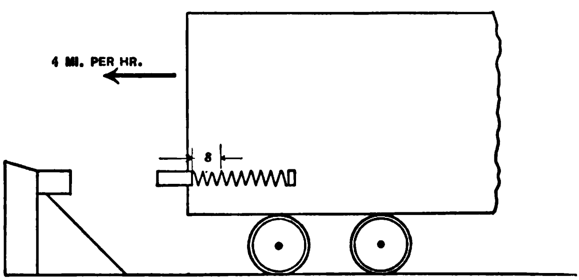


FIG. 164

113. Motion when Attractive Force Varies Inversely as the Square of the Distance. — This is the case of motion (Fig. 165) when two bodies in space are considered, since in such cases the attractive force varies directly as the product of their masses and inversely as the square of the distance between their centers of gravity. The same attraction holds between two opposite poles of magnets or between two bodies charged oppositely with electricity.

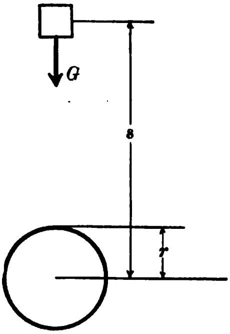


FIG. 165

Suppose the acceleration $= \dfrac{-k}{s^2}$ and that the initial velocity is zero.

Then, $\displaystyle\int_v^0 v\,dv = -\int_s^{s_0} \frac{k}{s^2}\,ds,$

so that $$\frac{ds}{dt} = -\sqrt{\frac{2k}{s_0}}\,\frac{\sqrt{s_0 s - s^2}}{s},$$

and $$t = \sqrt{\frac{s_0}{2k}}\left[\sqrt{s_0 s - s^2} - \frac{s_0}{2}\operatorname{vers}^{-1}\frac{2s}{s_0} + \frac{\pi s_0}{2}\right].$$

The time required to reach the center of attraction from the position of rest is obtained by putting $s = 0$.

This gives $t = \dfrac{\pi}{\sqrt{k}}\left(\dfrac{s_0}{2}\right)^{\frac{3}{2}}.$

It is seen that when $s = 0$, the velocity is infinite, and therefore the body approaches the center of attraction with increasing velocity and passes through the center, to be retarded on the other side until it reaches a distance $-s_0$. The motion will be oscillatory.

If one of the bodies is the earth, of radius r, and the other is a body of weight G falling toward it, the equations just derived hold true. In this case it is possible to determine k. The attraction on the body at the surface of the earth is G, and at a distance s is F, so that $F = G\left(\frac{r^2}{s^2}\right)$. The acceleration is therefore $\frac{-F}{M} = -g\left(\frac{r^2}{s^2}\right)$. This gives k, then, equal to $r^2 g$.

Substituting these values in the above equation, we find

$$v = \sqrt{\frac{2\,gr^2}{s_0}}\,\frac{\sqrt{s_0 s - s^2}}{s}.$$

When $s = r$ at the earth's surface,

$$v = \sqrt{\frac{2\,gr}{s_0}}\,\sqrt{(s_0 - r)} = \sqrt{2\,gr}\sqrt{\left(1 - \frac{r}{s_0}\right)}.$$

If $$s_0 = \infty, \quad v = \sqrt{2\,gr}.$$

But this is a value of v that cannot be obtained, since s_0 cannot be infinite. So that the velocity is always less than $\sqrt{2\,gr}$. It is interesting to notice here that if a body were projected from the earth with a velocity greater than $\sqrt{2\,gr}$, it would never return, provided there were no atmospheric resistance. Substituting $g = 32.2$ and $r = 3963$ mi.,

$$v = 6.95 \text{ mi. per sec.}$$

This is the greatest velocity that a body could possibly acquire in falling to the earth, and a body projected upward with a greater velocity would never return (neglecting resistance).

If the body falls to the earth from a height h, the velocity acquired may be obtained from the foregoing by putting $s = r$ and $s_0 = h + r$; then

$$v = \sqrt{\frac{2grh}{r+h}}.$$

If h is small compared to r, this may be written, without serious error,

$$v = \sqrt{2gh},$$

which is the formula derived for a freely falling body in Art. 105.

Problem 246. A body of mass 10 lb. has an initial velocity of 50 f/s and moves in a medium which resists with a force proportional to the velocity and equal to 1 lb. when the velocity is 4 f/s. How far will the body move before the velocity is reduced to 10 f/s? Would the velocity ever become zero with that law of resistance? What would be the limiting value of the space passed over?

Problem 247. A body of weight W lb. is projected vertically upward with a velocity V_0. If the resistance of the air is equal to kv^2, prove that the height to which the body will rise is

$$h = \frac{W}{2kg} \log_e\left(1 + \frac{kV_0^2}{W}\right).$$

Problem 248. The body of the preceding problem falls again to the earth. Show that the velocity with which it reaches the earth is

$$V_0 / \sqrt{1 + \frac{k}{W} V_0^2}.$$

Problem 249. A man jumps from a balloon and acquires a velocity of 80 f/s before his parachute is fully opened. The man and parachute weigh 150 lb. The resistance of the air varies as the square of the velocity (approximately) and is 1 lb. per square foot of opposing surface when the velocity is 20 f/s. If the diameter of the parachute is 12 ft., what velocity will the man have after descending a further distance of 200 ft.? after 1000 ft.?

Problem 250. On a toboggan slide of constant slope assume the friction to be constant and the resistance of the air to be proportional to the square of the velocity. Derive the formula for the velocity in terms of the distance, starting from rest.

114. Relative Velocity. — When we speak of the velocity of a body, it is understood that we mean the velocity of the body relative to the earth, more particularly the point on the earth from which the motion is observed. Since the earth is in motion, it is evident that velocity as generally spoken of is not absolute velocity, and since there is nothing in the universe that is at rest, all velocities must be relative. In everyday life, however, we think of velocities referred to any point on the surface of the earth as being absolute.

Suppose two particles, A and B, to have the velocities V_a and V_b, as illustrated in Fig. 166. *The motion of B relative to A is obtained by regarding A as at rest and B as having a velocity composed of the actual velocity of B and the reversed velocity of A at each instant.* Thus, the vector V, which is the resultant of V_b and V_a reversed, is the velocity of B relative to A at the instant at which the velocities are V_a and V_b.

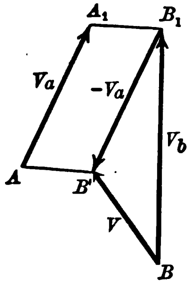


FIG. 166

This is made more evident in the case of constant velocities by the following consideration: if V_a and V_b are constant, then at the end of a unit of time A would be in the position A_1, and B in B_1, and their distance apart and direction from each other would be represented by the line A_1B_1, while if A re-

mained at rest, and B moved with a velocity composed of V_b and V_a reversed, the line AB' would represent the distance and direction of B from A. Since A_1B_1 and AB' are evidently equal and parallel, the relative positions of the two particles would be the same in one case as in the other.

As an illustration, consider the velocity of the wind relative to the sail of an ice boat. Let V_b and V_w be the velocities of the boat and wind respectively (Fig. 167) and SL the direction of the sail, which is assumed to be a smooth plane surface.

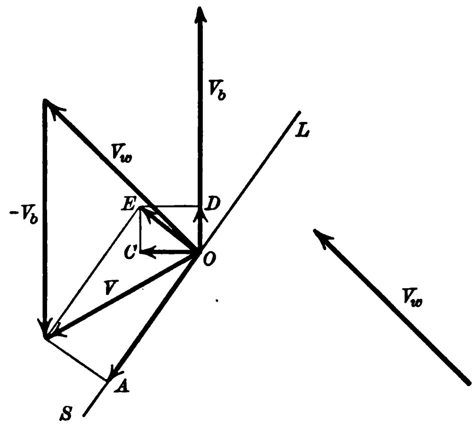


FIG. 167

Combining V_w with V_b reversed, the velocity V is obtained, which is the velocity of the wind relative to the boat. This velocity can be resolved into two components, OA and OE, along and perpendicular to the sail respectively. The component OA will have no effect on the boat on the assumption that the sail is a smooth surface. The component OE may be resolved into two components, OD and OC, respectively, along and perpendicular to the path of the boat. The vector OD then represents the component of the velocity of the wind which urges the boat forward.

From the figure it is clear that there may be a forward component of the wind's velocity on the boat even when the velocity of the boat is greater than the velocity of the wind. It is only necessary for the velocity of the wind relative to the boat to fall in front of the sail. Then as long as the forward pressure of the wind is greater than the resistance of the ice the velocity of the boat will increase. In the case of ice boats, where the resistance is small, the velocity of the boat may greatly exceed the velocity of the wind for high wind velocities.

Problem 251. A train is moving with a speed of 60 mi. per hour, and another train on a parallel track is going in the opposite direction with a speed of 40 mi. per hour. What is the velocity of the second train as observed by a passenger on the first?

Problem 252. A man in an automobile going at a speed of 40 mi. per hour is struck by a stone thrown by a boy. The stone has a velocity of 30 ft. per second and moves in a direction perpendicular to the direction of motion of the automobile. With what velocity does the stone strike the man?

Problem 253. A man attempts to swim across a river, $\frac{1}{4}$ mi. wide, which is flowing at the rate of 3 mi. per hour. If he can swim at the rate of 4 mi. per hour, what direction must he take in swimming in order to reach a point directly across on the opposite shore. What distance will he swim relative to the water? How long will it take for him to cross?

Problem 254. An ice boat is moving due north at a speed of 60 mi. per hour, and the wind blows from the southwest with a velocity of 40 mi. per hour. What is the apparent velocity and direction of the wind as observed by a man on the boat?

Problem 255. A man walks in the rain with a velocity of 4 mi. per hour. The raindrops have a velocity of 20 ft. per second in a direction making 60° with the horizontal. How much must the man incline his umbrella from the vertical in order to keep off the rain: (*a*)

when going against the rain, (*b*) when going away from the rain? If he doubles his speed, what change is necessary in the inclination of his umbrella in (*a*) and (*b*)?

Problem 256. The light from a star enters a telescope inclined at an angle of 45° with the surface of the earth. The velocity of light is 186,000 mi. per second and the earth (radius 4000 mi.) makes one revolution in 24 hr. What is the actual direction of the star with respect to the earth? This displacement of light due to the velocity of the earth and the velocity of light is known as *aberration of light.*

Problem 257. A locomotive is moving with a velocity of 40 mi. per hour. Its drive wheels are 80 in. in diameter. What is the tangential velocity of the upper point of the wheels with respect to the frame of the locomotive? What is the tangential velocity of the lowest point?

Problem 258. Show that if an ice boat were moving north and the wind had a velocity of 20 mi. per hour from the southwest, the maximum velocity that the boat could attain if there were no friction would be 38.6 mi. per hour if the sail were set at 30° with the direction of the boat's motion. What effect would increasing the angle between the sail and the boat's direction have in this case?

Problem 259. With the sail set at 30° with the direction of the boat's motion, in what direction could the boat go fastest if there were no friction? *Ans.* When the direction of the boat makes an angle of 60° with the direction of the wind.

Problem 260. Given that the angle which the sail makes with the direction of motion of the ice boat is 30°, the angle which the direction of the wind makes with the direction of the boat's motion is 45°, the area of the sail is 50 sq. ft., the resistance of the ice to the boat's motion is 16 lb., and given that the pressure of the wind normal to the surface of the sail varies as the square of the velocity (component of relative velocity normal to sail), and is 1 lb. per square foot of opposing surface when the velocity is 15 mi. per hour, find the maximum velocity that the boat can attain when the velocity of the wind is (*a*) 20 mi. per hour, (*b*) 30 mi. per hour.

Ans. (*a*) 14.6 mi. per hour. (*b*) 34.0 mi. per hour.

CHAPTER X

CURVILINEAR MOTION

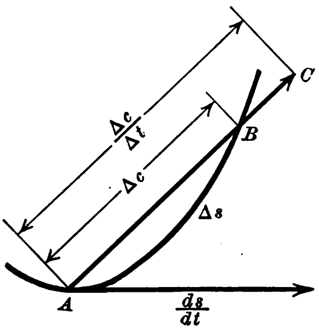


FIG. 168

115. Velocity. — Suppose a particle to be moving along the curve of Fig. 168, from A toward B. The average velocity of the particle between A and B is defined as the velocity the particle would have if it moved uniformly from A to B, *i.e.* along the chord AB with constant speed. The average velocity of the particle between A and B is therefore,

$$\text{average velocity} = \frac{\text{chord } AB}{\text{time spent between } A \text{ and } B}.$$

If the chord is represented by Δc and the time by Δt, then

$$\text{average velocity} = \frac{\Delta c}{\Delta t},$$

and is represented by a vector AC, of length $\frac{\Delta c}{\Delta t}$, laid off from A on AB. When the time Δt is made to approach the limit zero, the limiting value of the ratio $\frac{\Delta c}{\Delta t}$ is defined to be the velocity of the particle at the point A.

The limiting direction of the chord is the tangent at A. Hence, calling v the velocity of the particle at the point A,

$$v = \frac{dc}{dt}$$

and is represented by a vector of length $\frac{dc}{dt}$ laid off from A along the tangent in the direction of motion.

It is shown in calculus that if a single tangent to the curve at A exists, then the limit of the ratio of the arc to its chord, as the arc approaches the limit zero, is unity. Hence, if the arc AB is Δs, then

$$\frac{ds}{dc} = 1,$$

and

$$v = \frac{dc}{dt}\frac{ds}{dc} = \frac{ds}{dt}.$$

The velocity of a particle moving along a curved path is therefore represented at any point of the path by a vector equal to the value of $\frac{ds}{dt}$ laid off on the tangent to the path at the given point, and in the direction of the motion.

116. Acceleration in a Curved Path.—Let the velocities at A and B (Fig. 169 (a)) be v and $v + \Delta v$ respectively. Lay off the vectors representing these velocities from the same point A (Fig. 169 (b)).

The vector CD, from the end of v to the end of $v + \Delta v$, represents the change in velocity in the time Δt. The ratio of this vector to the time Δt is the average acceleration for the time Δt. Thus the vector CE represents the average acceleration of the particle between A and B.

The limiting value of this average acceleration as Δt approaches zero as a limit is defined to be the acceleration of the particle at A. Letting v_x and v_y be the projections

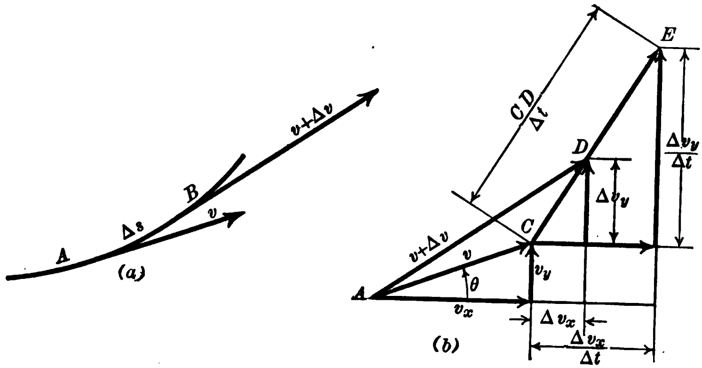


FIG. 169

of v upon rectangular axes, then the projections of CD on these axes are Δv_x and Δv_y and the projections of CE are $\frac{\Delta v_x}{\Delta t}$ and $\frac{\Delta v_y}{\Delta t}$. Hence

$$CE = \sqrt{\left(\frac{\Delta v_x}{\Delta t}\right)^2 + \left(\frac{\Delta v_y}{\Delta t}\right)^2},$$

and calling a the acceleration of the particle at A,

$$a = \sqrt{\left(\frac{dv_x}{dt}\right)^2 + \left(\frac{dv_y}{dt}\right)^2}.$$

If a_x and a_y are the components of a along the axes, then

$$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}.$$

Now $$v_x = v \cos\theta \text{ and } v_y = v \sin\theta$$

where θ is the angle which the tangent to the curve at A makes with the x-axis. Therefore

$$a_x = \cos\theta\frac{dv}{dt} - v\sin\theta\frac{d\theta}{dt},\quad a_y = \sin\theta\frac{dv}{dt} + v\cos\theta\frac{d\theta}{dt}.$$

Since $$v = \frac{ds}{dt},\quad \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

A value of $\frac{d\theta}{dt}$ may be found as follows: Draw a normal to the curve at A (Fig. 170). On this normal there is a point O', the center of a circle tangent to the curve at A and passing through B. Let r' be the radius of this circle and $\Delta\theta'$ the angle subtended at O' by the arc AB. As B is made to approach A as a limit the tangents to the curve and the circle at B take the same limiting position and hence the ratio $\frac{\Delta\theta}{\Delta\theta'}$ approaches 1 as a limit. The ratio of the arcs, $\Delta s'$ of the circle, and Δs of the curve, has the limit 1, since they have the same chord and the limit of the ratio of each arc to its chord is 1.

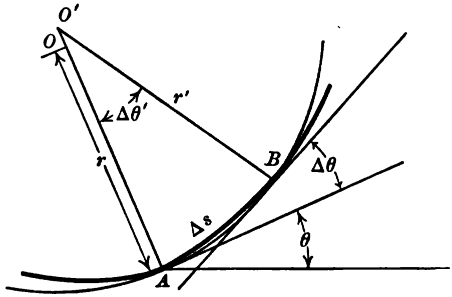


FIG. 170

As the point B is made to coincide with A the point O' takes some limiting position, O, which is called the *center of curvature* of the curve at A. If r is the value of OA, then limit $r' = r$.

Now $$r'\Delta\theta' = \Delta s', \text{ or } \frac{\Delta\theta'}{\Delta s'} = \frac{1}{r'}.$$

Therefore $$\frac{d\theta'}{ds'} = \frac{1}{r}, \text{ or since } \frac{d\theta}{d\theta'} = 1 \text{ and } \frac{ds}{ds'} = 1,$$

$$\frac{d\theta}{ds} = \frac{1}{r}, \text{ and } \frac{d\theta}{dt} = \frac{d\theta}{ds}\frac{ds}{dt} = \frac{1}{r}\frac{ds}{dt} = \frac{v}{r}.$$

The quantity r is the *radius of curvature* of the curve at A.

Substituting the values found for $\frac{dv}{dt}$ and $\frac{d\theta}{dt}$, the values of a_x and a_y become

$$a_x = \cos\theta\frac{d^2s}{dt^2} - \frac{v^2\sin\theta}{r},$$

$$a_y = \sin\theta\frac{d^2s}{dt^2} + \frac{v^2\cos\theta}{r}.$$

117. Tangential and Normal Components of the Acceleration.— The above values of a_x and a_y hold for any rectangular axes. Let the x-axis be taken along the tangent at A and the y-axis along the normal. Then $\cos\theta = 1$, and $\sin\theta = 0$ at the given point, and the values of a_x and a_y become respectively

$$a_t = \frac{d^2s}{dt^2}, \quad a_n = \frac{v^2}{r}.$$

The normal force and tangential force may now be written,

$$\textbf{Normal force} = \frac{Mv^2}{r};$$

$$\textbf{Tangential force} = M\frac{dv}{dt}.$$

For all curves except the circle r, the radius of curvature varies from point to point. In the circle, however, it is the radius, and is therefore constant. In this case the normal force is usually called the *centripetal force.*

Problem 261. A ball weighing 2 lb. is attached to one end of a string 5 ft. long, the other end of which is attached to a fixed point. The ball is pulled aside and released. When the string makes an angle of 30° with the vertical, the velocity of the ball is 10 f/s. Find the normal and tangential components of the acceleration and the acceleration in magnitude and direction at that instant. Find also the tension in the string. (Notice that the force acting toward the center on the body is the difference between the tension in the string and the component of the weight along the direction of the string.)

118. Uniform Motion in a Circle. — A body moving with constant speed, v, in the circumference of a circle is acted upon by one force, the normal or centripetal force, and this equals $\frac{Mv^2}{r}$. That this is true is evident when it is remembered that the tangential velocity is constant, thus making the tangential acceleration zero. An illustration of uniform motion in a circle is seen in the case of the simple governor shown in Fig. 171. When the speed is constant, then α, h, and r are constant. Let T be the tension in the rod supporting the ball, then, since there is no vertical motion $\Sigma Y = 0$, so that $T \cos \alpha = G$.

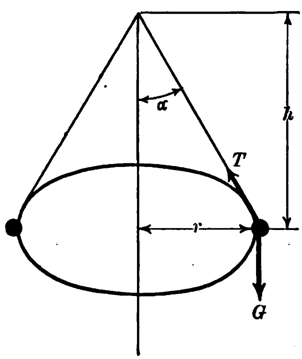


FIG. 171

Considering the normal force, we have $T \sin \alpha = \frac{Mv^2}{r}$, so that $\tan \alpha = \frac{v^2}{gr}$. From these equations T may be found for any values of α and r.

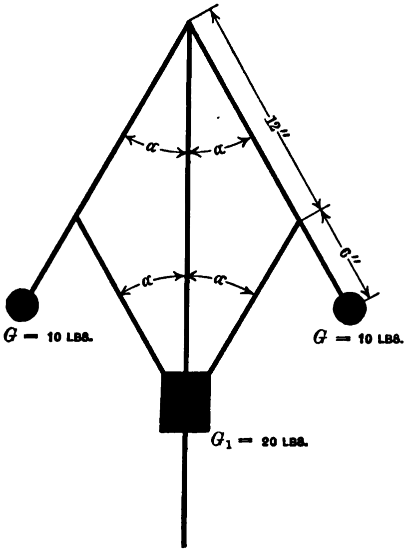


FIG. 172

Problem 262. The weighted governor shown in Fig. 172 is rotated at such a speed that $\alpha = 30°$. Find the forces acting on the longer rods and the stress in the shorter rods. The connections are all pin connections.

Problem 263. A type of swing is shown in Fig. 173. A revolving central post supported by wires A and B carries six cars G, each suspended from cross arms D by means of cables 50 ft. long. When the swing is at rest, the cars hang vertically and $\alpha = 0$; as the speed of rotation increases, α becomes larger. Suppose the car and its load of four passengers to weigh 1000 lb., and the speed to be such that $\alpha = 30°$; find the tension in the cables supporting the cars. Assume that a single car is carried by one cable.

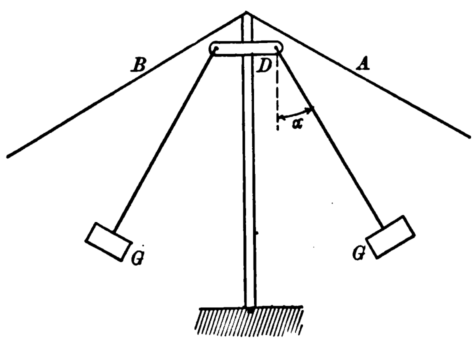


FIG. 173

Problem 264. The same principle that has been seen to hold for motion in a circle enables us to solve a problem that comes up in railroad work. When a train goes around a curve, it is desirable to have the outer rail raised sufficiently so that the

wheel pressure will be normal to the rails. It is really the same problem as Problem 263, where the sustaining cable is replaced by a track. (See Fig. 174.) Let r be the radius of curvature, v the velocity of the car, of weight G. Show that the superelevation of the outer rail is given by $\tan \alpha = \frac{v^2}{gr}$, and so, approximately,

$$h = \frac{dv^2}{gr},$$

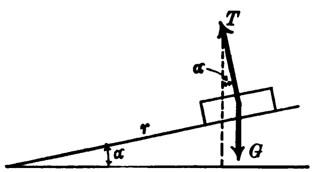


FIG. 174

where d is the distance between the rails in feet, v the velocity in feet per second, g is 32.2, r is the radius of curvature in feet, and h is the superelevation of the outer rail in feet.

Problem 265. Show that, using $d = 4.9$ ft., this height may be expressed, approximately, as follows:

$$h = \frac{v_1^2}{3\,r},$$

where h and r are in feet and v_1 is the velocity in miles per hour.

Problem 266. Find the superelevation of the outer rail on a curve of radius 2000 ft. for speeds of 20 mi. per hour, 40 mi. per hour, 60 mi. per hour.

Problem 267. What would the superelevation need to be for the above speeds on a curve of radius 500 ft.?

119. Motion without Friction along Any Curve in a Vertical Plane. — Let a particle slide down the smooth curved track (Fig. 175), starting from rest at the point A distant y_0 above a

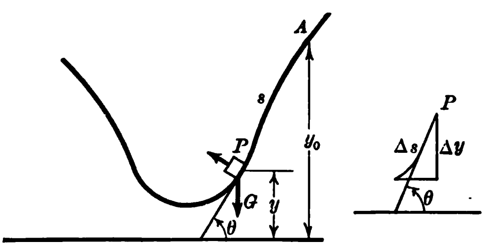


FIG. 175

horizontal line of reference. When in the position P, the forces acting on the particle are the weight G, acting vertically downward, and the force exerted by the track, which is normal to the track since there is no friction. The tangential force is then $G \sin \theta$, and hence for descending motion,

$$\frac{G}{g}\frac{d^2s}{dt^2} = G \sin \theta,$$

where s is the length of the curve AP measured from A toward P.

Now $\qquad \sin \theta = -\frac{dy}{ds}$ and also $\frac{d^2s}{dt^2} = v\frac{dv}{ds}.$

Therefore $\qquad v\frac{dv}{ds} = -g\frac{dy}{ds}.$

Integrating, $\qquad \frac{v^2}{2} = -gy + C.$

When $\qquad y = y_0, v = 0$; therefore $C = gy_0$.
Therefore

$$v^2 = 2g(y_0 - y).$$

Hence, *the velocity acquired by a body in sliding down a smooth track is the same at any point as if the body had fallen freely through the same vertical distance.*

The *time* of descent depends upon the form of the curve.

Problem 268. Show that the equation

$$vdv = -gdy$$

holds when the body is ascending as well as when descending, and then prove that the body would rise on the track to the level from which it started.

Problem 269. A body of weight G starts from rest at the top of a smooth circular track of radius R in a vertical plane. Find the velocity, acceleration, and force exerted by the track on the body when it has traveled through 60°, 90°, 150°, 180°, 210° of arc.

Problem 270. With what velocity would a body have to be projected from the lowest point of a smooth circular vertical track in order that its velocity would just keep it from falling from the track at the top?

Problem 271. If the body of the preceding problem were prevented from leaving the track, what velocity at the lowest point would just carry it around the track?

Problem 272. In the centrifugal railway (Fig. 176), friction neglected, what would have to be the ratio of h to h' so that the car would not leave the track at A, starting from rest at the height h?

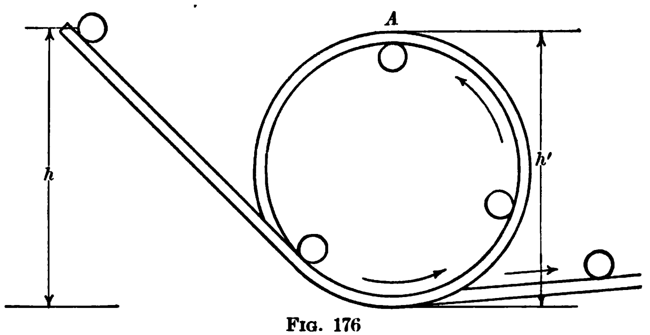


FIG. 176

Problem 273. If the car in the preceding problem were prevented from leaving the track, what would be the ratio of h to h' to just carry the car past A?

Problem 274. A particle slides from rest from any point on a smooth sphere. Show that it will leave the sphere when it has descended vertically through one third of its original vertical distance above the center.

Problem 275. A locomotive weighing 175 tons moves in an 800-ft. curve with a velocity of 40 mi. per hour. Find the horizontal pressure on the rails, if they are on the same horizontal.

Problem 276. If the velocity of the earth was 18 times what it actually is, show that the force of gravity would not be sufficient to keep bodies on the earth near the equator. Take the radius of the earth as 4000 mi., and assuming the above conditions, find at what latitude the body would just remain on the earth.

Problem 277. A pail containing 5 lb. of water is caused to swing in a vertical circle at the end of a string 3 ft. long. Find the velocity of the pail at the highest point so that the water will remain in the pail. Find also the velocity of the pail at the lowest point.

120. Simple Circular Pendulum. — The simple circular pendulum consists of a weight G suspended by a string without weight, of length l, in such a way that it is free to move in a circle in a vertical plane due to the action of gravity. (See Fig. 177.) Let B be such a position of the pendulum that its height above the horizontal is h, and C any other position designated by the coördinates x and y. Let the weight be G and the tension in the string T. These are the only forces acting on the body. The only forces that can produce motion in the circle are those that are tangent to the circle

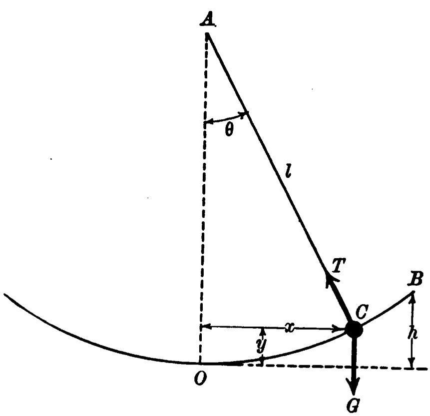


FIG. 177

OB. The force T is normal to the circle and so has no tangential component. The force G has a tangential component $-G\sin\theta$. The equation of motion is, therefore,

$$vdv = ads = -g\sin\theta ds,$$

where $\theta = \frac{\text{arc } OC}{l} = \frac{s}{l}$, where s denotes distance along the curve.

Since $s = l\theta$, $ds = ld\theta$,

and hence $\qquad vdv = -lg\sin\theta d\theta. \qquad (1)$

Approximate Solution. The integration of this equation leads to a complicated relation between the angle and the time. An approximate solution can be obtained for small angles of oscillation by replacing $\sin\theta$ by θ. Equation (1) then becomes

$$vdv = -lg\theta d\theta.$$

Integrating, $\qquad v^2 = -lg\theta^2 + C.$

Let $v = 0$ when $\theta = \theta_1$; then $C = lg\theta_1^2$.

$$\therefore v^2 = lg(\theta_1^2 - \theta^2).$$

$$\therefore v = \sqrt{lg}\sqrt{\theta_1^2 - \theta^2}.$$

For the body descending s decreases as t increases and therefore $\frac{ds}{dt}$ is negative, and

$$\frac{ds}{dt} = \frac{ld\theta}{dt} = -\sqrt{lg}\sqrt{\theta_1^2 - \theta^2}$$

or

$$\frac{d\theta}{\sqrt{\theta_1^2 - \theta^2}} = -\sqrt{\frac{g}{l}}\cdot dt.$$

Integrating, $\quad \sin^{-1}\frac{\theta}{\theta_1} = -\sqrt{\frac{g}{l}}t + C_1.$

Let $t = 0$ when $\theta = \theta_1$; then $\sin^{-1} 1 = C_1$.

From the angles whose sines are 1 choose $\frac{\pi}{2} = \sin^{-1} 1 = C_1$.

Then
$$\sqrt{\frac{g}{l}} \cdot t = \frac{\pi}{2} - \sin^{-1}\frac{\theta}{\theta_1}. \qquad (2)$$

As the pendulum swings from the highest point on the right to the point on the left on the same level (Art. 119), *i.e.* to where $\theta = -\theta_1$, the value of t continually increases. Equation (2) shows that $\sin^{-1}\frac{\theta}{\theta_1}$ must continually decrease during this time. Therefore $\sin^{-1}\frac{\theta}{\theta_1}$ must decrease in this interval from $\frac{\pi}{2}$ to $\sin^{-1}(-1)$, which therefore can only be $-\frac{\pi}{2}$. Hence if t is the time of a single oscillation,

$$\sqrt{\frac{g}{l}}t = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

or
$$t = \pi\sqrt{\frac{l}{g}}.$$

This value of t is called the period of vibration.

General Solution. For large vibrations the above approximate solution is not sufficiently accurate.

Integrating equation (1),

$$vdv = -gl \sin\theta d\theta,$$

we obtain
$$\frac{v^2}{2} = gl\cos\theta + C.$$

When
$$v = 0,\ \theta = \theta_1.$$

$$\therefore\ C = -gl\cos\theta_1$$

and
$$v^2 = 2gl(\cos\theta - \cos\theta_1).$$

$$\therefore\ l\frac{d\theta}{dt} = -\sqrt{2gl(\cos\theta - \cos\theta_1)}$$

or
$$\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_1}} = -\sqrt{\frac{2g}{l}}\,dt.$$

Since $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$, this equation may be written

$$\frac{d\theta}{\sqrt{\sin^2\frac{\theta_1}{2} - \sin^2\frac{\theta}{2}}} = -2\sqrt{\frac{g}{l}}\,dt.$$

If t_1 is the time that it takes for the pendulum to descend from $\theta = \theta_1$ to $\theta = 0$, then

$$\int_{\theta_1}^{0} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_1}{2} - \sin^2\frac{\theta}{2}}} = -2\sqrt{\frac{g}{l}}\int_0^{t_1} dt = -2\sqrt{\frac{g}{l}}\cdot t_1$$

or
$$t_1 = \frac{1}{2}\sqrt{\frac{l}{g}}\int_0^{\theta_1} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_1}{2} - \sin^2\frac{\theta}{2}}}.$$

The time of vibration, t, is double this, or

$$t = \sqrt{\frac{l}{g}}\int_0^{\theta_1} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_1}{2} - \sin^2\frac{\theta}{2}}}.$$

Change the variable by the substitution

$$\sin\frac{\theta}{2} = \sin\frac{\theta_1}{2}\sin\phi.$$

Then
$$\frac{1}{2}\cos\frac{\theta}{2}\,d\theta = \sin\frac{\theta_1}{2}\cos\phi\,d\phi,$$

and
$$t = 2\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \sin^2\frac{\theta_1}{2}\sin^2\phi}}.$$

This integral is an *elliptic integral.* The expression can be integrated only by expanding into a series and integrating term by term.

Thus, letting $k = \sin\frac{\theta_1}{2}$,

$$t = 2\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}(1 - k^2\sin^2\phi)^{-\frac{1}{2}}d\phi$$

$$= 2\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\left(1 + \frac{1}{2}k^2\sin^2\phi + \frac{1\cdot 3}{1\cdot 2}\frac{1}{2^2}k^4\sin^4\phi + \frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\frac{1}{2^3}k^6\sin^6\phi + \cdots\right)d\phi.$$

From the reduction formula

$$\int \sin^m x dx = -\frac{\cos x \sin^{m-1}x}{n} + \frac{m-1}{m}\int \sin^{m-2} x dx$$

there follows

$$\int_0^{\frac{\pi}{2}} \sin^m x dx = \frac{m-1}{m}\int_0^{\frac{\pi}{2}} \sin^{m-2} x dx = \frac{(m-1)(m-3)\cdots 3\cdot 1}{m(m-2)\cdots 4\cdot 2}\frac{\pi}{2},$$

and hence

$$t = \pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right],$$

where $k = \sin\frac{\theta_1}{2}.$

Problem 278. Using the series, compute correct to three figures the time of vibration of a simple pendulum of length 3 ft. when the pendulum swings through 20°. Compare with the time obtained from $t = \pi\sqrt{\frac{l}{g}}.$

Problem 279. Compute the time of vibration of a simple pendulum of length 4 ft. when swinging through an angle of 80°.

Problem 280. A pendulum vibrates seconds at a certain place and at another place it makes 60 more vibrations in 12 hours. Compare the values of g at the two places.

Problem 281. Show that in the simple pendulum for any angle of swing the value of the tension in the string is

$$T = G\left(\frac{l + 2h - 3y}{l}\right),$$

where h and y have the meaning given them in Art. 120.

Hence show that the tension is equal to the weight when the weight has descended through one third the original vertical distance.

Problem 282. The weight of a chandelier is 300 lb., and the distance of its center of gravity from the ceiling is 16 ft. Neglecting the weight of the supporting chain, find how much the tension in the chain will be increased if the chandelier is set swinging through an angle of 8°, measured at the ceiling.

121. Cycloidal Pendulum. — It has been found that a pendulum may be obtained whose period of vibration is constant by allowing the string to wrap itself around a cycloid as shown in Fig. 178. The pendulum hangs from the point A. AB and AC are cycloidal guides around which the string wraps as the pendulum swings. This causes the length of the pendulum to continually change and the pendulum "bob" to move in another cycloidal curve COB. The equation of this curve referred to the axes x and y is

$$x = \frac{l}{4}\text{vers}^{-1}\left(\frac{4y}{l}\right) + \sqrt{\frac{ly}{2} - y^2}.$$

In Art. 119 it was seen that $v^2 = 2g(h - y)$ represented the velocity of a body moving in a vertical curve when

only the force of gravity and a force normal to the path of the curve acted. We may make use of the equation in

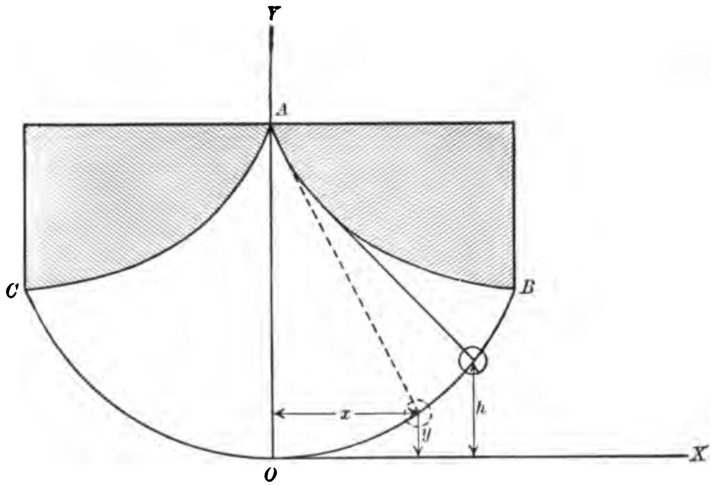


FIG. 178

this case, since the same conditions exist. We may write

$$dt = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2g(h-y)}}, \quad \text{since} \quad v = \frac{ds}{dt}.$$

From the equation of the curve, we find

$$dx = \sqrt{\frac{l - 2y}{2y}} \cdot dy,$$

so that
$$\int_0^t dt = \sqrt{\frac{l}{4g}} \int_h^0 \left(\frac{-dy}{\sqrt{hy - y^2}}\right),$$

taking the negative sign, since t is a decreasing function of y.

Therefore $$t = \sqrt{\frac{l}{4g}}\left[\text{vers}^{-1}\frac{2y}{h}\right]_0^h = \pi\sqrt{\frac{l}{4g}}.$$

The whole time of vibration is twice this value, so that the time of vibration

$$t = \pi\sqrt{\frac{l}{g}}.$$

This expression is independent of h, so that all vibrations are made in the same time. The motion is therefore isochronal.

Problem 283. Using the calculus formula for radius of curvature,

$$r = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

show that the radius of curvature of the cycloid of the above article at any point is

$$r = \sqrt{l(l - 2y)},$$

and hence show that the tension in the string for any position is

$$T = \frac{G(l + 2h - 4y)}{\sqrt{l(l - 2y)}},$$

where G is the weight of the pendulum bob.

Problem 284. Assuming a value of h in terms of l, say $h = \frac{l}{4}$, plot the curve representing the tension in terms of y. For what position of the pendulum is the tension a maximum?

Show that for any swing the maximum tension is

$$T = G\left(\frac{l + 2h}{l}\right).$$

Problem 285. A particle slides from rest down an inverted cycloid. Prove that its vertical velocity is greatest when it has de-

scended through one half the original vertical distance above the lowest point of the curve. What is the vertical velocity at that point?

122. Motion of Projectile in Vacuo. — A method, slightly different from the preceding, of dealing with a problem of curvilinear motion, is illustrated in the present article. It is desired to find the path taken by a body projected with a velocity v_0 at an angle of elevation α, when the resistance of the air is neglected. (See Fig. 179.)

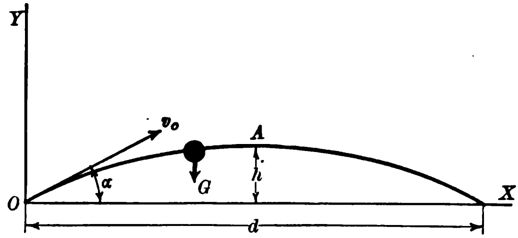


FIG. 179

Let the point of projection be taken as origin and the x-axis horizontal. Then, since there is no horizontal force acting on the body, $a_x = 0$, so that

$$\frac{d^2x}{dt^2} = 0$$

and
$$\frac{dx}{dt} = \text{constant} = v_0 \cos \alpha,$$

therefore
$$x = v_0 \cos \alpha(t).$$

In a similar way we know that the vertical acceleration $a_y = -g$, since the only force acting is G.

Then,
$$\frac{d^2y}{dt^2} = -g$$

and
$$\frac{dy}{dt} = -gt + \text{constant}.$$

This equation may be rewritten

$$v_y = -gt + \text{constant}.$$

To determine this constant of integration, we put $t = 0$,

and $$v_y = v_0 \sin\alpha;$$

therefore $$\frac{dy}{dt} = -gt + v_0 \sin\alpha$$

and $$y = -\tfrac{1}{2}gt^2 + v_0 \sin\alpha(t).$$

Eliminating t between the equations in x and y, we get

$$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$

as the equation of the path of the projectile. This is evidently a parabola, with its axis vertical.

Range. To find the range or horizontal distance d we put $y = 0$; then $x = 0$ and $x = d$,

so that $$d = \frac{v_0^2 \sin 2\alpha}{g}.$$

From this it is clear that the greatest range is given when $\alpha = 45°$, since then $d = \frac{v_0^2}{g}$.

The Greatest Height. The greatest height to which the projectile will rise is found by putting $x = \frac{v_0^2 \sin 2\alpha}{2g}$ in the equation of the curve and solving for y. This gives

$$h = \frac{v_0^2 \sin^2\alpha}{2g},$$

and the angle that gives the greatest height is $\alpha = 90°$. For this case $h = \frac{v_0^2}{2g}$. This is the case that has already

been considered under the head of a body projected vertically upward.

Problem 286. A fire hose delivers water with a nozzle velocity v_0, at an angle of elevation α. How high up on a vertical wall, situated at a distance d' from the nozzle, will the water be thrown? It should be said that water thrown from a nozzle in a non-resisting medium takes a parabolic path and follows the same laws as projectiles.

Problem 287. What at least must be the nozzle velocity of water thrown upon a burning building, 200 ft. high, the angle of elevation of the curve being 60°?

Problem 288. The muzzle velocity of a gun is 500 ft. per second. Find its greatest range when stationed on the side of a hill which makes an angle of 10° with the horizontal: (*a*) up the hill, (*b*) down the hill. If the hillside is a plane, prove that the area commanded by the gun is an ellipse, of which the gun is a focus.

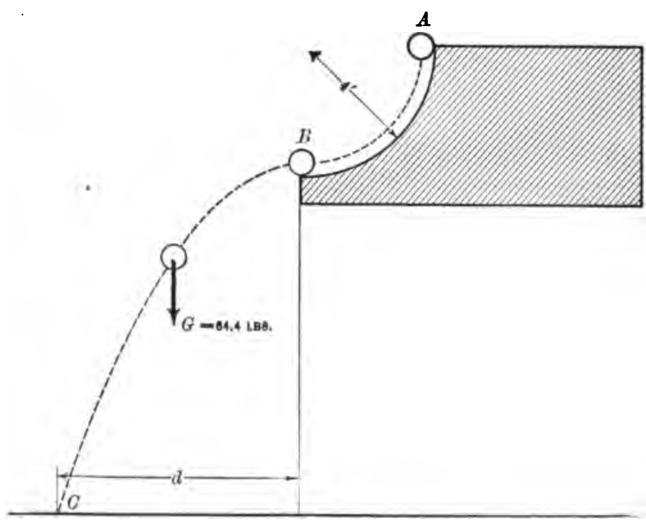


FIG. 180

Problem 289. From the foot of a plane inclined β to the horizontal a projectile is fired at an angle α to the horizontal up the plane. Prove that the range on the inclined plane is

$$\frac{2\,v_0^2 \cos\alpha \sin(\alpha - \beta)}{g \cos^2\beta}.$$

Problem 290. Prove that the initial velocity v_0 is the same as that of a body falling freely from the directrix of the parabolic path to the point O on the curve. Show that the velocity of the body at any point on the curve is the same as would be acquired in falling freely from the directrix to that point. (See Fig. 179.)

Problem 291. A ball whose weight is 64.4 lb., shown in Fig. 180, starts from rest at A and rolls without friction in a circular path to the point B, where it is projected from the circular path horizontally. Find (*a*) the velocity at B, (*b*) the equation of its path after leaving B, and (*c*) the distance d from a vertical through B, where it strikes a horizontal 10 ft. below B.

Problem 292. If the body in Problem 291 had moved along a straight line from A to B and was then projected, find, as in the preceding problem, (*a*), (*b*), and (*c*).

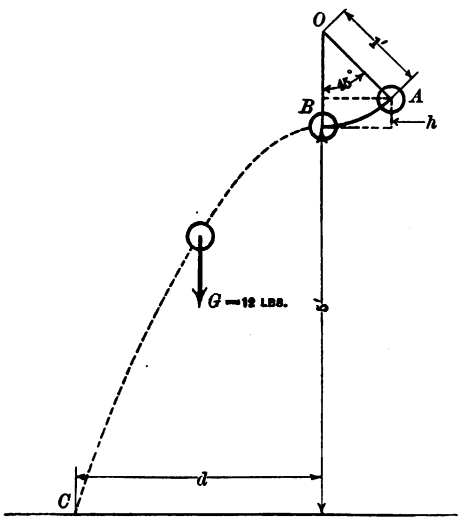


Fig. 181

Problem 293. A body whose weight is 12 lb. swings as a circular pendulum, as shown in Fig. 181, from A to B, when the string breaks.

Find (*a*) the velocity at *B*, (*b*) the equation of its path after leaving *B*, and (*c*) the distance *d* where it strikes a horizontal 5 ft. below *B*.

Problem 294. The muzzle velocity of a gun situated at a height of 300 ft. above a horizontal plane is 2000 ft. per second. Find the area of plane covered by the gun.

Problem 295. A projectile is fired from a mortar gun with an initial velocity of 25 mi. a minute. What is the maximum range on the horizontal, neglecting air resistance?

123. Motion of Projectile in Resisting Medium. — It was found by Rollins and others (see Encyclopædia Britannica — "Gunnery") that the formula for projectiles in vacuo did not hold when the projectile moved in the atmosphere. That is, that the path followed by the projectile was not parabolic, but on account of the resistance of the atmosphere the range was much less than that given by the parabola.

A formula constructed by Hélie, empirically modifying the parabolic formula, is

$$y = x \tan \alpha - \frac{gx^2}{2 \cos^2 \alpha}\left(\frac{1}{v_0^2} + \frac{kx}{v_0}\right),$$

where $k = 0.0000000458 \dfrac{d^2}{w}$, d being the diameter of the projectile in inches, and w its weight in pounds. This gives the simplest formula for roughly constructing a range table.

Professor Bashforth of Woolrich found, from a series of experiments made by him, that for velocities between 900 and 1100 ft. per second the resistance varied as v^6, for velocities between 1100 and 1350 ft. per second the resistance varied as v^3, and for velocities above 1350 ft. per second the resistance varied as v^2.

In addition to the resistance of the air other factors tend to change the path of the projectile from the parabolic form, viz. the velocity of the wind and the rotation of the projectile itself. Most projectiles are given a right-handed rotation, and this causes them to bear away to the right upon leaving the gun. This is called *drift*. Correction is made for drift and wind velocity upon firing.

Problem 296. Taking $v_0 = 1000$ f/s, $\alpha = 45°$, $d = 6$ in., $w = 150$ lb., find the range from Hélie's formula. Plot to the same scale the curve followed by the projectile and the parabola it would follow if there were no resistance.

Problem 297. With the projectile of the preceding problem and $v_0 = 1200$ f/s, find the angle of elevation to strike a point 200 ft. high, distant 1000 ft. horizontally, (*a*) using Hélie's formula, (*b*) using the parabola.

124. Motion in a Twisted Curve. — If a particle moves along a curve in space, it follows, as in the case of a two-dimensional curve, that the velocity of the particle at any point of the curve is in the direction of the tangent to the curve at that point, and its value is

$$v = \frac{ds}{dt}. \quad \text{(Fig. 182.)}$$

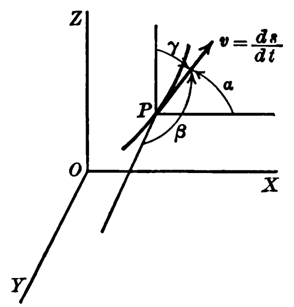


FIG. 182

If the tangent makes angles α, β, γ, with the coördinate axes, then

$$v_x = \frac{dx}{dt} = v \cos \alpha, \quad v_y = \frac{dy}{dt} = v \cos \beta, \quad v_z = \frac{dz}{dt} = v \cos \gamma.$$

If a is the acceleration of a particle when at the point P

and a_x, a_y, a_z, the components of a parallel to the axes, then

$$a_x = \frac{d^2x}{dt^2} = \frac{d(v\cos\alpha)}{dt},$$

$$a_y = \frac{d^2y}{dt^2} = \frac{d(v\cos\beta)}{dt},$$

$$a_z = \frac{d^2z}{dt^2} = \frac{d(v\cos\gamma)}{dt},$$

or

$$a_x = \cos\alpha\frac{dv}{dt} - v\sin\alpha\frac{d\alpha}{dt},$$

$$a_y = \cos\beta\frac{dv}{dt} - v\sin\beta\frac{d\beta}{dt},$$

$$a_z = \cos\gamma\frac{dv}{dt} - v\sin\gamma\frac{d\gamma}{dt}.$$

The sum of the projections of a_x, a_y, a_z on the tangent is the component of the acceleration along the tangent; *i.e.*

$$\begin{aligned} a_t &= a_x\cos\alpha + a_y\cos\beta + a_z\cos\gamma \\ &= (\cos^2\alpha + \cos^2\beta + \cos^2\gamma)\frac{dv}{dt} \\ &\quad - v\left(\sin\alpha\cos\alpha\frac{d\alpha}{dt} + \sin\beta\cos\beta\frac{d\beta}{dt} + \sin\gamma\cos\gamma\frac{d\gamma}{dt}\right). \end{aligned}$$

But $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, and therefore

$$-2\left(\cos\alpha\sin\alpha\frac{d\alpha}{dt} + \cos\beta\sin\beta\frac{d\beta}{dt} + \cos\gamma\sin\gamma\frac{d\gamma}{dt}\right) = 0.$$

Therefore $$a_t = \frac{dv}{dt}.$$

If a particle slides without friction down any space curve, it can be shown as in the corresponding case of the

plane curve (Art. 109) that the velocity attained in descending a vertical distance h from rest is given by

$$v^2 = 2gh.$$

In the actual case of a body sliding along a curved chute, friction on the side and bottom of the chute and air resistance reduce the velocity below that given by the above formula.

As an illustration consider the motion of a body along a helical chute with vertical axis (Fig. 183).

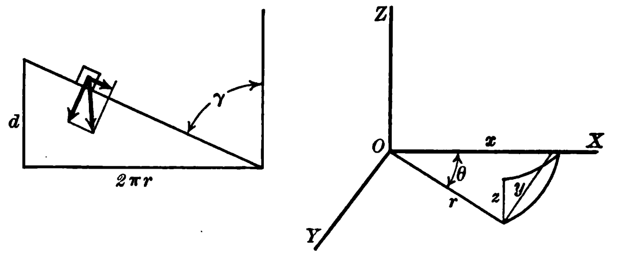


FIG. 183

If the z-axis is the axis of the helix and the curve passes through a point on the x-axis, then

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$z = \frac{d}{2\pi}\theta,$$

where d is the *pitch* of the helix, or vertical distance traveled by the generating point per turn. Developing one thread of the helix, it is seen that the tangent makes a *constant* angle γ with the z-axis such that

$$\tan\gamma = \frac{2\pi r}{d}.$$

During the motion of the body the chute exerts a force towards the axis of the curve of value $\frac{mv^2}{r}$. This force causes a retarding friction force along the tangent of value $c\frac{mv^2}{r}$, where c is a constant depending upon the roughness of surface of chute and body. In addition there is a tangential friction force due to the force exerted by the bottom of the chute on the body. The component of the weight perpendicular to the bottom of the chute is $W \sin \gamma$ and the friction force caused by it is $cW \sin \gamma$. Therefore the total retarding force is

$$c\left(W \sin \gamma + \frac{Wv^2}{gr}\right).$$

The component of the weight along the tangent is $W \cos \gamma$. Hence the net accelerating force along the tangent is

$$\begin{aligned} F_t &= W\left(\cos \gamma - c \sin \gamma - c\frac{v^2}{gr}\right) \\ &= \frac{Wc}{gr}(b^2 - v^2), \text{ where } b^2 = \frac{gr}{c}\cos \gamma - gr \sin \gamma. \end{aligned}$$

Hence the acceleration along the tangent is

$$a_t = \frac{F_t}{m} = \frac{c}{r}(b^2 - v^2),$$

or

$$\frac{dv}{dt} = \frac{c}{r}(b^2 - v^2).$$

Therefore

$$\frac{dv}{b^2 - v^2} = \frac{c}{r}\,dt.$$

Integrating,

$$\frac{1}{2\,b} \log \frac{b + v}{b - v} = \frac{c}{r}t + C_1.$$

If the body starts from rest, $v = 0$ when $t = 0$.

$$\therefore\ C_1 = \frac{1}{2b}\log 1 = 0.$$

Therefore
$$\frac{b+v}{b-v} = e^{\frac{2bc}{r}t}.$$

Solving for v,

$$v = b\,\frac{e^{\frac{2bc}{r}t} - 1}{e^{\frac{2bc}{r}t} + 1}$$

$$= b\left(1 - \frac{2}{e^{\frac{2bc}{r}t} + 1}\right)$$

This formula shows that v remains always less than the constant quantity b.

On account of the velocity not exceeding a fixed value this form of chute is useful in sending packages from one floor of a building to another.

Problem 298. Given a helical chute of radius 3 ft. and pitch 2 ft., find the velocity of a small body descending from rest after 5 sec., after 10 sec., if the value of c is .05.

CHAPTER XI

ROTARY MOTION

125. Angular Velocity and Angular Acceleration of a Particle. — If a particle moves in any plane curve, the rate at which a line joining the particle to a fixed point in the plane is turning is called the *angular velocity of the particle with respect to that point.*

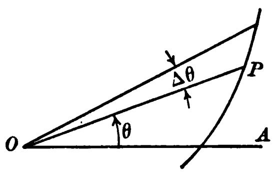


FIG. 184

Thus, if θ is the angle which the line joining the particle makes with a fixed line through O (Fig. 184), the angular velocity, ω, of the particle with respect to O is

$$\omega = \frac{d\theta}{dt}.$$

The rate of change of the angular velocity is called the *angular acceleration of the particle with respect to the point.*

Thus, if α is the angular acceleration of the particle with respect to O,

$$\alpha = \frac{d\omega}{dt}, \text{ or } \alpha = \frac{d^2\theta}{dt^2}.$$

The angular acceleration may also be written in the form

$$\alpha = \frac{d\omega}{d\theta}\frac{d\theta}{dt}, \text{ or } \alpha = \omega\frac{d\omega}{d\theta}.$$

The angular velocity and angular acceleration of a particle moving in a given path with a given speed depend upon the point to which they are referred.

Problem 299. Prove that if a particle moves around a circle, the angular velocity of the particle with respect to a point on the circumference is equal at any instant to one half the angular velocity of the particle with respect to the center of the circle, and hence that the angular velocity of the particle at any instant is the same with respect to all points on the circumference.

Show also that a like statement holds for angular acceleration.

Problem 300. Show that if a particle have constant angular acceleration with respect to any point, the following formulæ hold:

$$\alpha = \frac{\omega - \omega_0}{t}.$$

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2.$$

$$\omega^2 - \omega_0^2 = 2\alpha\theta,$$

where θ is the angle turned through in the time t, ω_0 the initial angular velocity, and ω the angular velocity at the instant t. Compare these formulæ with those of Art. 104.

Problem 301. A flywheel making 100 revolutions per minute is brought to rest in 2 min. Find the angular acceleration α and the angle θ turned through before coming to rest.

Problem 302. A flywheel is at rest, and it is desired to bring it to a velocity of 300 radians per minute in $\frac{1}{2}$ min. Find the angular acceleration α necessary and the number of revolutions required. What is the angular velocity ω at the end of 10 sec.?

126. Motion in a Circle.—If a particle is moving in a circle, the linear velocity at any instant is

$$v = \frac{ds}{dt}. \qquad \text{(Art. 115.)}$$

But here $$s = r\theta,$$

where s is the arc passed over and θ the angle subtended by this arc at the center.

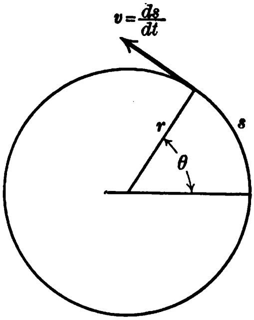


FIG. 185

$$\therefore \frac{ds}{dt} = r\frac{d\theta}{dt},$$

or $$v = r\omega.$$

Taking the second derivative,

$$\frac{d^2s}{dt^2} = r\frac{d^2\theta}{dt^2}.$$

From Art. 117 $\frac{d^2s}{dt^2}$ is the tangential component, a_t, of the linear acceleration of the point, and hence

$$a_t = r\alpha.$$

Problem 303. Prove that for a particle moving in a circle with angular velocity ω about the center,

$$\frac{dx}{dt} = -y\omega, \quad \frac{dy}{dt} = x\omega,$$

where x and y are the coördinates of the position of the particle referred to rectangular axes through the center.

Problem 304. The balance wheel of a watch goes backward and forward in $\frac{1}{2}$ sec. The angle through which it turns is 180°. Find the greatest angular acceleration and the greatest angular velocity, given that the angular acceleration varies directly as the angular displacement and is toward the neutral position of the spring.

Problem 305. Suppose the flywheel in Problem 302 to be 6 ft. in diameter. After arriving at the desired angular velocity, what is the tangential velocity of a point on the rim? What has been the tangential acceleration of this point, if constant?

Problem 306. Assume that the angular acceleration of a particle varies inversely as the square of the angle turned through; find the relation between ω and θ, and t and θ.

127. Angular Velocity with Respect to an Axis. — If a particle moves along any space curve, its angular velocity with respect to any fixed axis is the rate at which a line passing through the particle and the axis, perpendicular to the axis, is turning about the axis; or it is the angular velocity of the projection of the moving particle on a plane perpendicular to the axis with respect to the foot of the axis on that plane (Fig. 186).

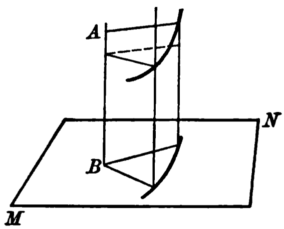


FIG. 186

Problem 307. The coördinates of a moving particle are

$$x = r\cos ct, \quad y = r\sin ct, \quad z = kt,$$

where r, c, and k are constants and t is the time. Prove that the angular velocity about the z-axis is c. What is the curve traced out ?

128. Plane Motion of a Body. — A body is said to have *plane motion* when each point of the body moves in a fixed plane. The plane in which the center of gravity of the body moves will be called the *plane of motion.*

The rate of turning of any fixed line of the body in the plane of motion is called the *angular velocity of the body.* It should be noted that the idea of rotation about an axis does not enter into this definition of angular velocity of a body.

129. Relation between the Velocities of Points of a Body having Plane Motion. — Let A and B (Fig. 187) be two

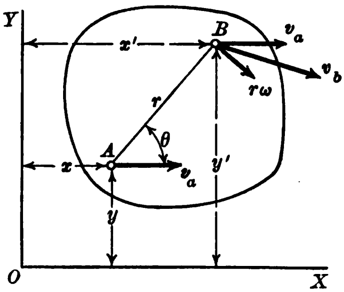


FIG. 187

points of the body in the plane of motion. Choose axes OX and OY in the plane of motion.

If (x, y) and (x', y') are the coördinates of A and B respectively and θ is the angle which AB makes with the x-axis, then

$$x' = x + r\cos\theta, \tag{1}$$

$$y' = y + r\sin\theta, \tag{2}$$

where r is the constant distance AB.

Therefore
$$\frac{dx'}{dt} = \frac{dx}{dt} - r\sin\theta\frac{d\theta}{dt}, \tag{3}$$

$$\frac{dy'}{dt} = \frac{dy}{dt} + r\cos\theta\frac{d\theta}{dt}. \tag{4}$$

If A were a point fixed in space, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ would be zero and the components of the velocity of B would become

$$\frac{dx'}{dt} = -r\sin\theta\frac{d\theta}{dt},$$

$$\frac{dy'}{dt} = r\cos\theta\frac{d\theta}{dt}.$$

If A is in motion, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are the components of the velocity of A. It therefore follows from equations (3) and (4) that the velocity of B is compounded of the velocity of A and the velocity of B relative to A regarded as at rest.

Hence if v_a is the velocity of A at any instant and ω the angular velocity of the body at that instant, the velocity of B is the resultant of v_a and $r\omega$, the latter being at right angles to AB (Fig. 187).

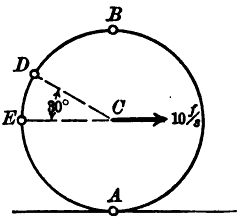


FIG. 188

Problem 308. A wheel of radius 3 ft. rolls along a straight line, the velocity of the center being 10 f/s (Fig. 188). Find the velocities of the points A, B, D, and E shown in the figure. Find the velocity of A relative to B, of E relative to A, of D relative to E.

Problem 309. Show that the velocity of B in Fig. 187 is

$$v_b = \sqrt{v_a^2 + r^2\omega^2 + 2\,r\omega v_a \sin\theta}.$$

130. Relation between the Accelerations of Points of a Body having Plane Motion. — Taking the derivatives of the equations (3) and (4) of Art. 129, it follows that the acceleration of a point in the plane of motion of a body having plane motion is composed of the acceleration of any other point in the plane of motion and the acceleration of the first point relative to the second regarded as at rest.

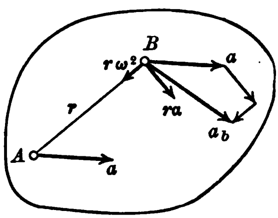


FIG. 189

Thus if a body in plane motion has angular velocity ω and angular acceleration α, the acceleration a_b of a point B is made up of a, the acceleration of A, $r\omega^2$ along BA, and $r\alpha$ perpendicular to AB (Fig. 189).

Problem 310. Find the acceleration of each of the points A, B, D, E, in Problem 308, given that the velocity of the center is constant. Represent these velocities by vectors.

Problem 311. If the point C in Problem 308 has an acceleration of $2\ f/s^2$ in the direction of motion at the given instant, find the acceleration of each of the points A, B, D, E.

Problem 312. Show that for a wheel rolling along a straight line the tangential velocity and acceleration of any point of the circumference relative to the center regarded as at rest are the same in magnitude as the velocity and acceleration of the center of the wheel.

Problem 313. A locomotive drive wheel 6 ft. in diameter rolls along a level track. Find the greatest tangential acceleration and the greatest normal acceleration of any point on the tread, (*a*) when the constant velocity v with which the wheel moves along the track is 60 mi. per hour, (*b*) when the engine is slowing down uniformly and has a velocity of 30 mi. per hour at the end of 3 min., (*c*) when the engine is starting up uniformly and has a velocity of 30 mi. per hour at the end of 5 min.

131. Instantaneous Axis of Rotation for Plane Motion. — Let A and B be two points of a body having plane motion, lying in the plane of motion. If the body does not have a motion of translation only, A and B can always be chosen so that their directions of motion are not parallel. Through A draw in the plane of motion a line, AO, perpendicular to the direction of motions of A. Any point of the body on AO, if it is in motion at all, must be moving perpendicular to AO, since any two points of the body remain always the same distance apart. Likewise, a point on OB, which is perpendicular to the direction of motion of B, must, if moving at all, be moving

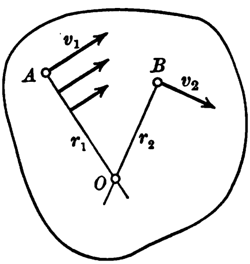


FIG. 190

perpendicular to OB. The point of intersection, O, of OA and OB, must therefore be at rest, since it cannot at the same time be moving perpendicular to OA and OB.

The line through O, perpendicular to the plane of motion, is called the *instantaneous axis of rotation.* Every point on this axis has velocity zero at the given instant, and the motion of the body is at the instant one of rotation about this line.

If the figure in motion is a plane figure moving in the plane in which it lies, the point O is called the *instantaneous center of rotation.*

The path traced by the instantaneous center is called the *centrode.*

If $OA = r_1$, $OB = r_2$, and the velocities of A and B are respectively v_1 and v_2, then the angular velocity of the body is

$$\omega = \frac{v_1}{r_1} = \frac{v_2}{r_2}.$$

It follows that $v_1 : v_2 = r_1 : r_2$.

Problem 314. A straight line of length a moves with its ends on two axes at right angles to each other. Prove that the centrode is a quadrant of a circle of radius a and center at the intersection of the two axes.

Problem 315. A connecting rod is 4 ft. long, and the crank-pin circle is 1 ft. in radius. Sketch the centrode of the connecting rod for one complete stroke.

Problem 316. The diameter of the wheel outlined in Fig. 191 is 6 ft. The velocities of the points B and D are as indicated by the vectors. Find (a) the instantaneous center, (b) the magnitude

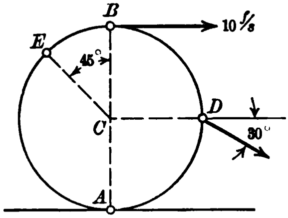


Fig. 191

of the velocity of the point D, (c) the angular velocity of the wheel, (d) the velocity of the point E, (e) the velocity of slipping of the point A of the wheel.

132. Simultaneous Rotation of a Body about Two Intersecting Axes. — Suppose a body rotating about an axis OA (Fig. 192) with an angular velocity ω_1, relative to a framework attached to OA, and let OA and the attached framework be at the same instant rotating about an axis OB with angular velocity ω_2.

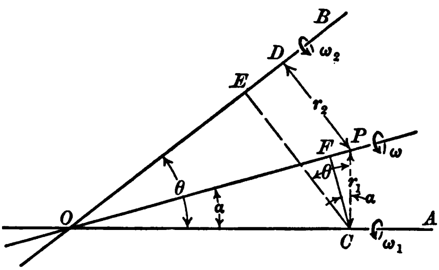


FIG. 192

If P be a point of the body, or a point of space thought of as moving with the body, in the plane containing OA and OB, between OA and OB, and distant r_1 and r_2, respectively, from these lines, the velocity of P due to the two rotations is

$$v = r_1\omega_1 - r_2\omega_2.$$

If P is so chosen that r_1 and r_2 satisfy the relation $r_1 : r_2 = \omega_2 : \omega_1$, the velocity of P is zero. The locus of all such points is clearly a straight line through O. Hence there is a straight line, OP, in the plane of the two intersecting axes of rotation which is at rest at the given instant, and about which the body is rotating at that instant.

To find the angular velocity, ω, of the body about this instantaneous axis OP, consider the motion of the point

C on OA. Draw CE perpendicular to OB and CF perpendicular to OP.

The point C has angular velocity ω_2 about OB, and hence its velocity is $EC\omega_2$. Its angular velocity about OP is ω and hence its velocity is $FC\omega$.

$$\therefore\ FC\omega = EC\omega_2.$$

Now $\quad FC = OC \sin \alpha$ and $EC = OC \sin \theta$.

$$\therefore\ \omega \sin \alpha = \omega_2 \sin \theta. \qquad (1)$$

Again, $\quad FC = r_1 \cos \alpha$, and $EC = r_2 + r_1 \cos \theta$.

Therefore $\quad r_1\omega \cos \alpha = r_2\omega_2 + r_1\omega_2 \cos \theta.$

But $\quad r_2\omega_2 = r_1\omega_1,$

and hence $\quad \omega \cos \alpha = \omega_1 + \omega_2 \cos \theta. \qquad (2)$

Squaring and adding equations (1) and (2),

$$\omega^2 = \omega_1^2 + \omega_2^2 + 2\,\omega_1\omega_2 \cos \theta. \qquad (3)$$

Dividing (1) by (2),

$$\tan \alpha = \frac{\omega_2 \sin \theta}{\omega_1 + \omega_2 \cos \theta}. \qquad (4)$$

Equations (3) and (4) are exactly the equations that would be obtained by regarding ω_1 and ω_2 as vectors and finding their resultant as in Fig. 193.

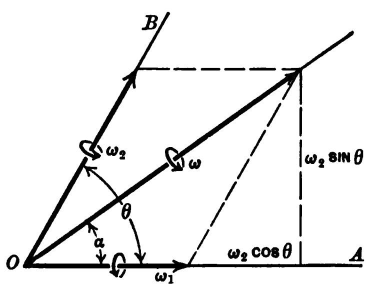


FIG. 193

Hence we may represent angular velocity of a body about a line by a vector laid off on that line, equal in length to the numerical value of the angular ve-

locity, and may combine simultaneous angular velocities about two intersecting lines by combining their vectors in the usual way. The resultant vector represents the value of the resultant angular velocity and coincides with the instantaneous axis of rotation of the body.

The vector representing an angular velocity is made to point opposite to the direction from which the rotation appears counter-clockwise to an observer looking along the axis of rotation.

It is evident at once, then, that the angular velocity of a body about a line may be regarded as made up of component angular velocities about two or more axes intersecting that line, obtained by using the parallelogram law of vectors. In particular a rotation about an axis may be regarded as made up of the simultaneous rotations about three axes at right angles to each other and intersecting the given line (Fig. 194).

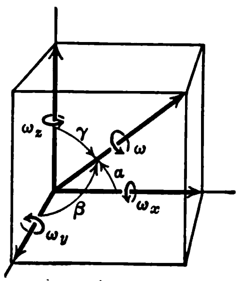


FIG. 194

If the angular velocity of the body about the line is ω, the angles which this line makes with the three rectangular axes, α, β, γ, and the angular velocities of the body about these axes ω_x, ω_y, ω_z respectively, then

$$\omega_x = \omega \cos \alpha, \qquad \omega_y = \omega \cos \beta, \qquad \omega_z = \omega \cos \gamma,$$

and
$$\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2.$$

Problem 317. Prove that the vector representation holds for simultaneous rotation about two parallel axes, *i.e.* that the resultant angular velocity of ω_1 and ω_2 is

$$\omega = \omega_1 + \omega_2,$$

and that the instantaneous axis of rotation divides the line joining the axes of ω_1 and ω_2 in the inverse ratio of ω_1 and ω_2. (See Fig. 195.)

Problem 318. A wheel is making 120 r. p. m. about a shaft and the shaft at the same time is making 90 r. p. m. about a line perpendicular to the axis of the shaft. Find the instantaneous axis of the shaft at any instant.

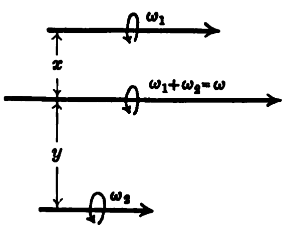


FIG. 195

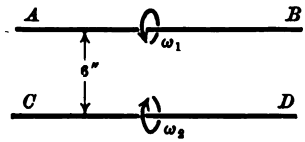


FIG. 196

Problem 319. A body has simultaneous rotations $\omega_1 = 4\,\pi$ rad./sec., $\omega_2 = 3\,\pi$ rad./sec., about the parallel lines of Fig. 196 in the directions indicated. Locate the instantaneous axis of rotation and find the angular velocity of the body about that axis.

133. Rotation of the Earth. Foucault's Pendulum. — Let ω be the angular velocity of the earth about its axis of rotation CP (Fig. 197). This angular velocity can be resolved into two components about rectangular axes CA and CB. If λ is the latitude of the position of A, the component of ω about CA is $\omega \sin \lambda$. If a plane containing CA could be fixed so as not to turn about CA, the rotation of the earth about CA would be indicated by the apparent rotation of the plane in the

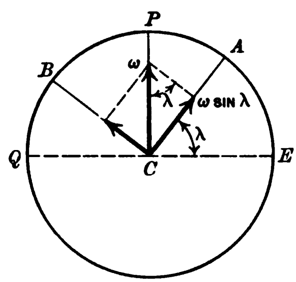


FIG. 197

opposite direction about CA, since our directions are measured with reference to the earth's surface.

Such a plane is the plane of vibration of a heavy weight suspended by a long wire.

Foucault was the first to demonstrate the rotation of the earth by this method.

Problem 320. Show that the rotation of the earth about a diameter through a point on the equator is zero.

Problem 321. Show that the time for the plane of the Foucault's pendulum to turn through 360° is $\frac{24}{\sin\lambda}$ hours, where λ is the latitude of the place.

Problem 322. The plane of a Foucault's pendulum is observed to turn through 26° in 2 hr. 20 min. Find the latitude of the place.

CHAPTER XII

WORK AND ENERGY

134. Definitions. — If a force, constant in magnitude and direction, acts at a fixed point of a body, the force is said to do work on the body when the point of application of the force has a displacement with a component in the direction in which the force acts.

The product of the force and the component of the displacement in the direction of the force is defined to be the work done on the body by the force in that displacement. If there is a displacement opposite to the direction of the force, work is said to be done against the force.

For example, in Fig. 198, let the forces P, R, G, N act upon the block as the block moves from the position A to

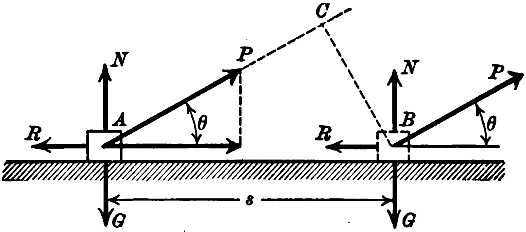


FIG. 198

the position B. If BC is perpendicular to the line of P, the displacement in the direction of P is AC and the work done by P on the block is $P \cdot AC$. If $AB = s$, then $AC = s \cos \theta$, and the work done by P is $Ps \cos \theta$. This

may be written $P \cos \theta \cdot s$. But $P \cos \theta$ is the component of P in the direction of the displacement of the point of application, and therefore the work done by P is equal to the product of the displacement of the point of application and the component of the force in the direction of the displacement.

Since N and G are at right angles to the displacement, no work is done by N or G.

Since the displacement is opposite to the direction of R, work is done against R, the amount being Rs.

If the body had moved from B to A during the action of the given forces, the work done by R would be Rs, and the work done against P would be $P \cos \theta \cdot s$.

If the point of application changes in the body, the work done on the body is the product of the force and the displacement, the displacement meaning the actual displacement of the force in space plus the relative displacement of the moving point of application past the initial point of the force vector in the direction opposite to the direction in which the force acts. If this relative displacement is in the direction in which the force acts, it must be subtracted from the displacement of the force in space.

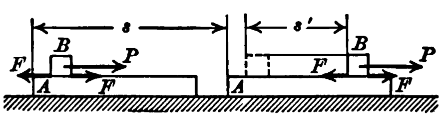


FIG. 199

For example, in Fig. 199 suppose a force P, applied to the block B, drags B along A and at the same time moves A. Let F be the forward force exerted by B on A. If when A moves forward a distance s, B moves relative to A a distance s', then the relative dis-

placement of the point of application past the initial point of the force is s' in the direction in which the force acts, and hence the work done by F on A is $F(s + s' - s')$ or Fs. On the other hand, a force F acts backward on B and the block B does work against this resistance to an amount equal to $F(s + s')$. Hence the total work done against the forces of resistance between the bodies is Fs', or the product of the force and the distance one body moves relative to the other.

135. Friction Forces.—Wherever the surfaces of two bodies come in contact and there is motion of one body along the other, or any force tending to produce such motion, each body exerts upon the other a force along their common surface, a force tending to prevent the relative motion of one surface along the other. This resisting force is known as *friction*. The laws of friction are discussed in a later chapter. In general the law of friction of unlubricated surfaces may be expressed by saying that the force of friction for two given surfaces is proportional to the normal pressure between the surfaces.

The ratio of the force of friction to the normal force is called the *coefficient of friction*.

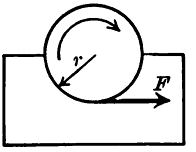


FIG. 200

Consider the work done against friction on an axle rotating in a fixed bearing (Fig. 200). Here the point of application of the friction force, F, changes in the body while the initial point of the force vector remains fixed in space. The displacement is then the dis-

placement of the outer points of the axle past the fixed point at which F acts. The work done on the axle by friction, in resisting its motion, is therefore the product of F and the distance through which a point on the circumference moves; or in turning through an angle θ the work done is $Fr\theta$.

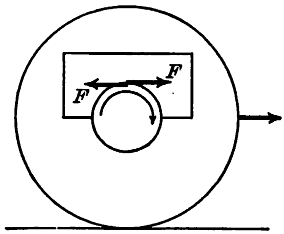


FIG. 201

If the bearing is also in motion, as in Fig. 201, then if the bearing moves forward s ft. while the axle rubs a distance s' ft. past the bearing, the work of F on the bearing is Fs, and the work done against F on the axle is $F(s+s')$, and hence the total work done against friction is Fs'.

136. Units of Work. — The unit of work involves the unit of space and the unit of force. If the distance is measured in feet and the force in pounds, the unit of work is the work done by a force of 1 lb. when the displacement in the direction in which the force acts is 1 ft.

This unit of work is called the *foot-pound*, designated by *ft.-lb.*

In the c. g. s. system the *erg* is defined as the work done by a force of 1 dyne when the displacement in the direction in which the force acts is 1 cm.

The *gram-centimeter* is the work done by a force of 1 gram working through a distance of 1 cm., etc.

137. Work of Components of a Force. — Let the constant force P (Fig. 202) acting on a body have the displacement

AB, or *s*. Resolve *P* into any two rectangular components, *X* and *Y*. In the given movement the displacement of *X* is *AC*, or *x*, and of *Y* is *AD*, or *y*. The work done by *P* in the given displacement is $P \cos \theta \cdot s$.

But $P \cos \theta$ is the projection of *P* on *AB* and is therefore equal to the sum of the projections of *X* and *Y* on *AB*; *i.e.*

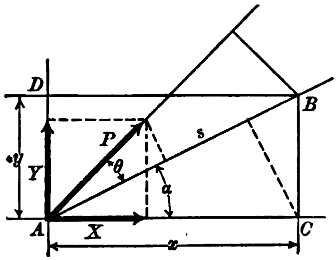


FIG. 202

$$P \cos \theta = X \cos \alpha + Y \sin \alpha.$$

$$\therefore \quad P \cos \theta \cdot s = Xs \cos \alpha + Ys \sin \alpha$$
$$= Xx + Yy.$$

Therefore, *the work done by a constant force in any displacement is equivalent to the work done by any rectangular components of the force in the same displacement.*

Problem 323. A weight of 20 lb. is dragged 50 ft. up a plane inclined 30° to the horizon by a constant force, $P = 25$ lb., acting at an angle of 15° to the plane. There is a retarding force, $R = 5$ lb., along the plane. Compute the work done by, or against, each force acting on the body. Find the work done by the horizontal and vertical components of *P* in the given displacement, and also the work done by the components of *W* perpendicular to and parallel to the given plane.

If the body started from rest, what velocity does it have at the end of the 50 ft.?

Problem 324. A block, *A*, weighing 50 lb., is pulled by a horizontal force of 32 lb. along a horizontal plane. A block, *B*, weighing 20 lb., rests on top of *A*. If the coefficient of friction between *A* and

the plane is $\frac{1}{5}$ and that between B and A is $\frac{1}{4}$, will B slip on A? If so, how far would B slip back on A when A has moved forward 10 ft.?

What work has been done against friction in this motion?

Problem 325. In the preceding problem what would the coefficient of friction between A and B have to be in order that B would just not slip? What then would be the total work done against friction when A has moved forward 10 ft.?

138. Work of a Variable Force. — If a force acting on a fixed point of a body changes in magnitude and direction and the point on which it acts moves along a curve, the

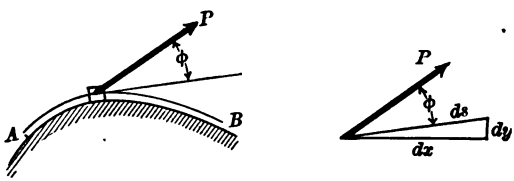


FIG. 203

work done in any displacement from A to B (Fig. 203) is defined to be

$$W_p = \int P \cos \phi ds,$$

the integration being taken from A to B, where ϕ is the angle at any point of the curve between the force and the tangent to the curve.

Since ds, dx, and dy are related just as s, x, and y, in Art. 137, we may write

$$W_p = \int X dx + \int Y dy,$$

or, if the curve is in three dimensions,

$$W_p = \int X dx + \int Y dy + \int Z dz.$$

Problem 326. Prove by integration that the work done against gravity in bringing the weight W from the position A to the position B (Fig. 204) along the quadrant of a circle is Wr.

Problem 327. Show from the definition,

$$\text{Work} = \int P \cos \phi ds,$$

that when a weight W is brought from any position along any curve to a position h ft. higher, the work done against the earth's attraction for the body is Wh.

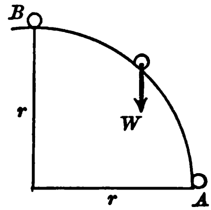


FIG. 204

139. Graphical Representation of Work. — Work has been defined as the product of a force and a distance. If the force be *uniform* and equal to P, and the body upon which it acts be moved through a distance a, the graphical representation of the work done by P is given by the area of a rectangle (Fig. 205) constructed on P and a as sides, since

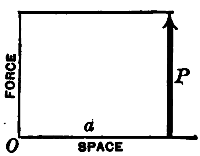


FIG. 205

$$W = Pa.$$

If the force P varies as the distance through which the body is moved along its line of action, we may represent the work by the area of the triangle as shown in Fig. 206. Let the force be zero when the motion begins, and let it be P_1 when the distance passed over along its line of action is OA. Then since the force varies as the distance, it is equal to P as shown in Fig. 206 for any intermediate distance OC. The total work

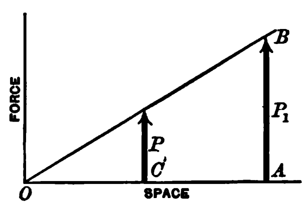


FIG. 206

done, then, in moving the body through a distance OA by the variable force P, which varies as the distance, is equal to $\frac{(P_1 OA)}{2}$. It is seen that this is the same as the work done by the average force $\frac{P_1}{2}$ acting through the distance OA. The resistance of a helical spring varies with the elongation or compression. The same law of variation holds for all elastic bodies.

Another variation of force with distance with which the engineer is frequently concerned, is the case where the force *varies inversely as the distance* through which the body is moved. In this case, if P is the force and S the distance, the relation between force and distance may be expressed,

$$PS = \text{const.}$$

But this represents the equilateral hyperbola. This will be made clearer by reference to the specific example of the expansion of steam in a steam cylinder. (See Fig. 207.) Up to the point of cut-off C, the steam pressure is the same as that in the boiler (practically), and is constant while the piston moves from O to C. At this point, the entering steam is cut off and the work done must be done by the

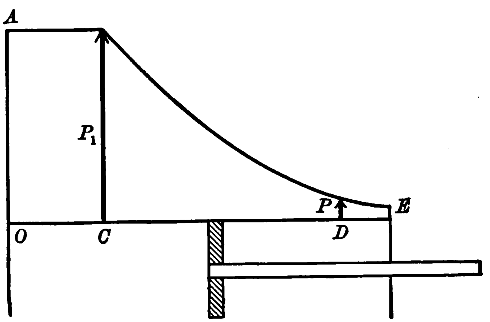


FIG. 207

expansion of the steam now in the cylinder. According to Mariotte's Law, the pressure varies inversely with the volume of steam; but since the cross section of the cylinder is constant, we may say that the pressure varies inversely as the distance. From the definition of work,

$$W_P = \int P ds,$$

it follows that the area under the curve represents the work done.

The curve obtained in practice representing the relation between the force and distance is shown in Fig. 208.

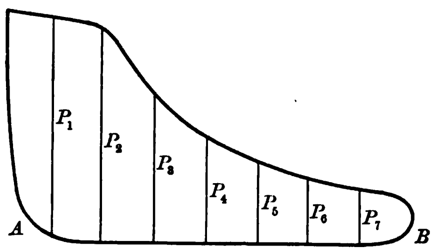


FIG. 208

The curve after cut-off is not a true hyperbola, and its area is determined by means of a planimeter or by Simpson's Rule.

140. Power.—The idea of work is independent of time. But for economical reasons it is necessary to take into consideration this element of time. We must know whether certain work has been done in an hour or ten hours. For such information a unit of the rate at which work is done has been adopted. This unit is called power. *Power is the rate of doing work. It is the ratio of the work done to the time spent in doing that work.*

The unit of power is the *horse power.* This has been taken as 550 ft.-lb. per second, or 33,000 ft.-lb. per minute.

Originally the idea of the rate of work was connected with the rate at which a good draft horse could do work. This value as used by Watt was 550 ft.-lb. per second. The horse power of a steam engine is mean effective pressure times distance traveled by the piston per second, divided by 550.

141. Energy. — *Energy is the capacity for doing work;* it is stored-up work. Bodies that are capable of doing work due to their position are said to possess *potential energy.* Bodies that are capable of doing work due to their motion are said to possess *kinetic energy.* A familiar example of potential energy is the energy possessed by a brick as it is in position on the top of a chimney. If the brick should fall, its energy at any instant would be called kinetic. When the brick strikes the ground, work is done in deforming the ground and brick, or perhaps even breaking the brick and even generating heat. The work done by the brick when it strikes is sufficient to use up all the energy that the brick had when it struck.

142. Conservation of Energy. — The kinetic energy of the brick spoken of in the last article was used up in doing work on the ground and air, and upon the brick itself, so that the kinetic energy that the brick possessed when it struck was used up. It was not, however, destroyed, but was transferred to other bodies, or into heat. Such transference is in accord with the well-known principle of the conservation of energy. This principle may be stated as follows: *energy cannot be created or destroyed.* The amount of energy in the universe is constant. This means that the energy given

up by one body or system of bodies is transferred to some other body or bodies. It may be that the energy changes its form into light, heat, or electrical energy.

Energy cannot be created or destroyed; it is, therefore, evident that such a thing as *perpetual motion* is impossible. Such a motion would involve the getting of just a little more energy from a system of bodies than was put into them.

143. Relation between Work and Energy. — Suppose the particle of mass M acted upon by any set of forces to move along the path from A to B (Fig. 209). Resolve the forces along and perpendicular to the curve at each point. Let F_t be the tangential component of the resultant of all forces acting on the particle. If a_t is the tangential component of the acceleration, then

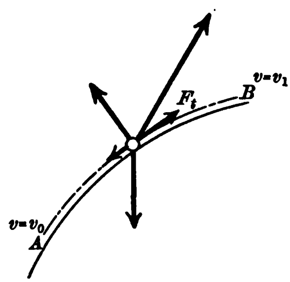


FIG. 209

$$F_t = M \cdot a_t.$$

But $$a_t = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}. \qquad \text{(Art. 117.)}$$

$$\therefore F_t ds = Mv \cdot dv.$$

The work done on the particle in going from A to B is

$$\int_A^B F_t ds. \qquad \text{(Art. 138.)}$$

If the velocity of the particle is v_0 at A and v_1 at B,

$$\therefore \int_A^B F_t ds = \int_{v_0}^{v_1} Mvdv = \tfrac{1}{2} M(v_1^2 - v_0^2),$$

or, **The work done on the particle** $= \frac{1}{2} M(v^2 - v_0^2)$. (1)

The kinetic energy of the particle has been defined as the work the particle can do due to its velocity. Suppose in Fig. 209 F_t is opposite to the direction of motion and the point B is such that the particle comes to rest there from the velocity v_0 at A. Then, from the definition,

$$\text{K. E. of particle at } A = \int_A^B F_t ds.$$

In this case a_t is negative, $\quad a_t = -v\dfrac{dv}{ds}.$

Therefore $$F_t = -Mv\frac{dv}{ds},$$

and $$\int_A^B F_t ds = -\int_{v_0}^0 Mvdv = \tfrac{1}{2}Mv_0^2.$$

Therefore the kinetic energy of a particle with velocity v is $\frac{1}{2}Mv^2$. Equation (1) of this article may then be expressed, *The work done on a particle in any displacement by the resultant of all the forces acting on the particle is equal to the gain in kinetic energy of the particle.*

For any body having a motion of translation only the proof applies as well as to a particle.

Problem 328. A car whose weight is 20 tons, having a velocity of 60 mi. per hour, is brought to rest by means of brake friction after the power has been shut off. If the tangential force of friction of 200 lb. acts on each of the 8 wheels, how far will the car go before coming to rest?

Problem 329. Suppose the car in the preceding problem to be moving at the rate of 60 mi. per hour when the power is shut off, what tangential force on each of the 8 wheels will bring the car to rest in one half a mile?

Problem 330. What is the kinetic energy of a river 200 ft. wide and 15 ft. deep, if it flows at the rate of 4 mi. per hour, the weight of a cubic foot of water being 62.5 lb.? What horse power might be developed by using all the water in the river?

Problem 331. The height of free fall in Niagara Falls is 165 ft. Assuming the velocity of the water at the top to be zero, what is its velocity at the foot? The flow is approximately 270,000 cu. ft. per second. How much kinetic energy is developed per second? What H. P. could be developed by using all the water? Counting the height of the fall, including rapids above and below, as 216 ft., what H. P. could be developed by using all the water?

NOTE. It is estimated that the total horse power of Niagara Falls, considering the fall as 216 ft., is 7,500,000. The Niagara Falls Power Company diverts a part of the volume of water above the rapids into their power plants, where it passes through a tunnel into the river below the falls. The turbines are 140 ft. below the water level, and each one is acted upon by a column of water 7 ft. in diameter. The estimated power utilized in this way is 220,000 horse power. The student should estimate the horse power of each turbine, assuming the water to fall from rest through 140 ft. For a full account of the power at Niagara Falls, the student is referred to Proc. Inst. C. E., Vol. CXXIV, p. 223.

Problem 332. A body whose weight is 32.2 lb. is pulled up a plane, inclined at an angle of 30° with the horizontal, by a horizontal force of 250 lb. The motion is resisted by a constant force of friction of 10 lb. acting along the plane. If it starts from rest, what will be its velocity after it has gone up a distance of 100 ft.?

Problem 333. The same body as that in the preceding problem is projected down the plane with a velocity of 5 ft. per second. How far will it go before coming to rest?

Problem 334. Solve Problem 240 by work and energy.

Problem 335. Solve Problem 242 by work and energy.

Problem 336. Solve Problem 243 by work and energy.

Problem 337. A body whose weight is 64.4 lb. falls freely from rest from a height of 5 ft. upon a 200-lb. helical spring. Find the compression in the spring.

Problem 338. A weight of 500 lb. is to fall freely from rest through a distance of 6 ft. The kinetic energy is to be absorbed by a helical spring. Specifications require that the spring shall not be compressed more than 2 in. Find the strength of the spring required.

Problem 339. Specifications state that it shall require 32,000 lb. to compress a helical spring 1½ in. What weight falling freely from rest through a height of 10 ft. will compress it one inch?

Problem 340. The draft rigging of a freight car shown in Fig. 210 is provided with two helical springs, one inside the other. The outside spring is a 10,000-lb. spring, and the inside a 5000-lb. spring. A car weighing 60,000 lb. is provided with such a draft rigging. While going at the rate of 1 mi. per hour it collides with a bumping post. How much will the springs be compressed?

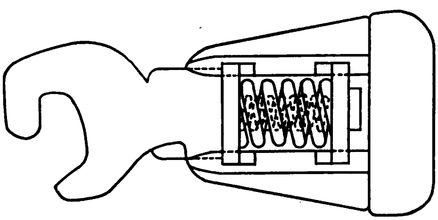


FIG. 210

Problem 341. The draft rigging in the preceding problem is attached to the first car of a freight train, consisting of 30 cars, each weighing 60,000 lb. How much will the springs of the first car be elongated if there is 10 lb. pull for each ton of weight when the speed is 40 mi. per hour? The speed is increased to 45 mi. per hour. How much will the springs be elongated if the resistance per ton at this speed is 12 lb.?

Problem 342. The Mallet compound locomotive (*Railway Age*, Aug. 9, 1907) is capable of exerting a draw-bar pull of 94,800 lb. According to the preceding problem, how many 60,000-lb. cars can be pulled at 45 mi. per hour? What strength of spring would be necessary for the first car, if the allowable compression is 1½ in.?

Here the work per revolution becomes (Fig. 246)

$$\textbf{Work per revolution} = \int_{r_1}^{r_2}\int_0^{2\pi} 2\,\pi r' \frac{fP}{\pi(r_2^2 - r_1^2)}\, r'\,dr'\,d\theta$$

$$= \frac{4\,\pi}{3}\,\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}\,fP.$$

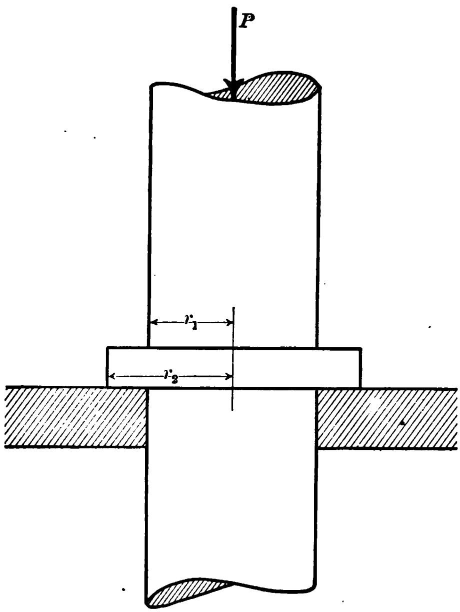


FIG. 246

(*c*) *Conical Pivot.* The conical pivots, illustrated in Fig. 247, do not usually fit into the step the entire depth of the cone. Let the radius of the cone at the top of the step be r_1, α half the angle of the cone, and let dP_1 be the normal pressure of the bearing on an elementary area whose horizontal projection is $r'\,dr'\,d\theta$.

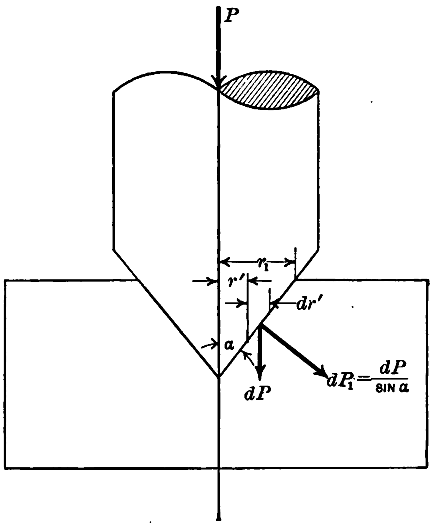


FIG. 247

The vertical component of dP_1 is equal to the vertical load on the horizontal area $r'dr'd\theta$ which is

$$dP = \frac{P}{\pi r_1^2} r'dr'd\theta.$$

$$\therefore\ dP_1 = \frac{Pr'dr'd\theta}{\pi r_1^2 \sin\alpha}$$

and the force of friction acting on this element is

$$dF = \frac{fPr'dr'd\theta}{\pi r_1^2 \sin\alpha}.$$

The total work per revolution done against friction is therefore

$$\textbf{Work per revolution} = \int_0^{r_1}\int_0^{2\pi} \frac{fP}{\pi r_1^2 \sin\alpha}\, 2\,\pi r'^2 dr' d\theta$$

$$= \frac{4}{3}\,\frac{\pi r_1 f P}{\sin\alpha}.$$

If $\alpha = \frac{\pi}{2}$, this value for work lost reduces to the work lost per revolution, in the case of the flat-end solid pivot. It is easily seen, since $\sin\alpha$ is less than unity, that if r_1 is nearly equal to r, the friction of the conical bearing is greater than the friction of the flat-end bearing. This might have been expected from the wedgelike action of the pivot on the step. It is also easily seen that r_1 may

be taken small enough so that the friction will be less than the friction of the flat pivot. The work lost due to friction in the case of the conical pivot will be equal to, greater, or less than, the work lost, due to friction in the case of the flat-end pivot, according as

$$r_1 \overset{=}{\underset{<}{>}} r \sin \alpha.$$

(*d*) *Spherical Pivot.* Suppose the end of the pivot is a spherical surface, as shown in Fig. 248. Let r be the radius of the shaft and r_1 the radius of the spherical surface; then the load per unit of area of horizontal surface is $\frac{P}{\pi r^2}$.

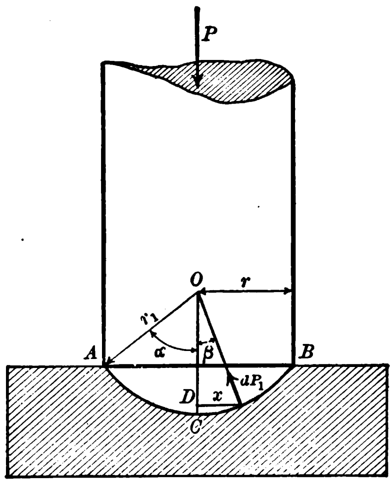


FIG. 248

The horizontal projection of any elementary ring of the bearing, of radius x, is $2\pi x dx$. The load on this area is

$$2\pi x dx\left(\frac{P}{\pi r^2}\right) = \frac{2 Pxdx}{r^2},$$

and the corresponding normal pressure is

$$dP_1 = \frac{2 Pxdx}{r^2} \sec \beta,$$

and the total work done against friction in one revolution is

$$\int_0^r \frac{4\pi f P x^2 dx}{r^2 \cos \beta}.$$

Here β is a variable such that

$$x = r_1 \sin \beta.$$

$$\therefore dx = r_1 \cos \beta d\beta,$$

and the expression for the work becomes

$$\textbf{Work per rev.} = \frac{4\,\pi f P r_1^3}{r^2}\int_0^{\alpha} \sin^2 \beta d\beta$$

$$= \frac{4\,\pi f P r_1^3}{r^2}\left(\frac{\alpha}{2} - \frac{1}{2}\sin\alpha\cos\alpha\right)$$

$$= 2\,\pi f P r_1\left[\frac{\alpha}{\sin^2\alpha} - \cot\alpha\right],$$

since $$r = r_1 \sin \alpha.$$

If the bearing is hemispherical, $\alpha = \frac{\pi}{2}$, and the work lost per revolution becomes

$$W = f\pi^2 Pr.$$

The friction of flat pivots is often made much less by forcing oil into the bearing, so that the shaft runs on a film of oil. In the case of the turbine shafts of the Niagara Falls Power Company (see Art. 143) the downward pressure is counteracted by an upward water pressure. In some cases the end of a flat pivot has been floated on a mercury bath. This reduces the friction to a minimum. (See *Engineering*, July 4, 1893.)

The Schiele pivot is a pivot designed to wear uniformly all over its surface. The surface is a tractrix of revolution ; that is, the surface formed by revolving a tractrix about its asymptote. Its value as a thrust bearing is not as great as was first anticipated. (See *American Machinist*, April 19, 1894.)

$$P(CA) = 2\,T_1(BA) - 2\,T_2(BA)$$
$$= 2\,BA(T_1 - T_2),$$

so that
$$T_1 - T_2 = \frac{P}{2}\frac{CA}{BA}.$$

The distances CA and BA are known, and P may be measured; the difference, then, $T_1 - T_2$, may always be obtained. The value $T_1 - T_2$ is then known and the horse power determined by the relation

$$\text{H.P.} = \frac{(T_1 - T_2)2\,\pi r n}{33{,}000},$$

where n is the number of revolutions per minute, and r is the radius of the machine pulley in feet.

168. Creeping or Slip of Belts. — A belt that transmits power between two pulleys is tighter on the *driving side* than it is on the *following side*. On account of this difference in tension and the elasticity of the material, the tight side is stretched more than the slack side. To compensate for this greater stretch on one side than on the other, the belt *creeps* or *slips* over the pulleys. This slip has been found for ordinary conditions to vary from 3 to 12 ft. per minute. The coefficient of friction when the slip is considered is about .27 (Lanza). It has also been found that the loss in horse power in well-designed belt drives, due to slip, does not exceed 3 or 4 per cent of the gross power transmitted, and that ropes are practically as efficient as belts in this respect. For an account of the experimental investigations on this subject the student is referred to Inst. Mech. Eng., 1895, Vols. 3–4, p. 599, and Trans. Am. Soc. M. E., Vol. 26, 1905, p. 584.

169. Coefficient of Friction of Belting. — The value of the coefficient of friction of belting depends not only on the slip but also upon the condition and material of the rubbing surfaces. Morin found for leather belts on iron pulleys the coefficient of friction $f = .56$ when dry, .36 when wet, .23 when greasy, and .15 when oily (Kent, "Pocket-Book"). Most investigators, however, including Morin, took no account of slip, so that the best value of f, everything considered, is that given in the preceding article (.27).

170. Friction of a Worn Bearing. — The friction of a bearing that fits perfectly is the friction of one surface sliding over another and is given by the equation

$$F = fN,$$

where F is the force of friction, f the coefficient of friction, and N is the total normal pressure on the bearing.

When, however, the bearing is worn, as is shown much exaggerated in Fig. 244, the friction may be somewhat

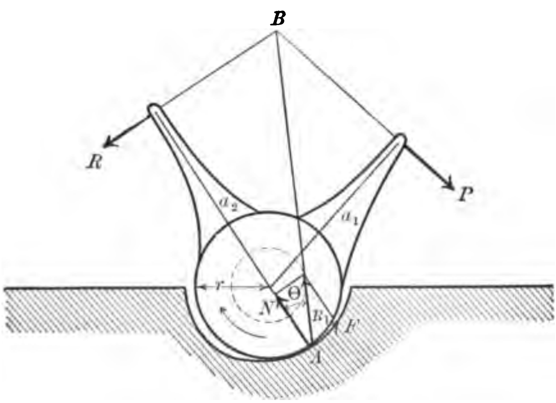


FIG. 244

different. When motion begins, the shaft will roll up on the bearing until it reaches a point A where slipping begins. If motion continues, slipping will continue along a line of contact through A. Let P be a force that causes the rotation, R a force tending to resist the rotation, and R_1 the reaction of the bearing on the shaft. There are only three forces acting on the shaft, so that P, R, and R_1 must meet in the point B. The direction of R_1 is accordingly determined. The normal pressure is $N = R_1 \cos \theta$, and the force of friction is

$$F = R_1 \sin \theta.$$

It is seen that θ is the angle of friction. The moment of the friction with respect to the center of the axle is

$$Fr = R_1 r \sin \theta.$$

If the axle is well lubricated, so that θ is small and $\sin \theta$ may be replaced by $\tan \theta = f$, the friction is

$$F = fR_1,$$

and the moment

$$Fr = fR_1 r.$$

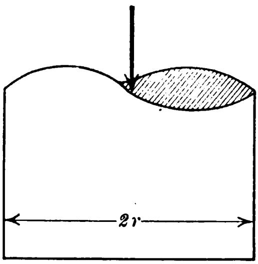
The circle tangent to AB of radius $r \sin \theta$ is called the *friction circle.* Since r and θ are known generally, this circle may be made use of in locating the point A.

The shaft will continue to rotate in the bearing so long as the reaction R_1 falls within the friction circle, and slipping will begin as soon as the direction of R_1 becomes tangent to the friction circle.

Problem 396. If the radius of the shaft is 1 in., $\theta = 4°$, $P = 500$ lb., $a_1 = 3$ ft., $a_2 = 2$ ft., angle between a_1 and a_2 is 100°, and

P and R are right angles to a_1 and a_2, what resistance R may be overcome by P when slipping occurs?

Problem 397. The radius of a shaft is 1 in., $R = 20$ lb., $P = 20$ lb., $a_1 = 3$ ft., and $a_2 = 2$ ft. What force of friction will be acting at the point A, when the angles between P and a_1 and R and a_2 are right angles? What must be the value of the coefficient of friction?



171. Friction of Pivots. — The friction of pivots presents a case of sliding friction, so that the force of friction F equals the coefficient of friction times the normal pressure. That is,

$$F = fN.$$

(*a*) *Flat-end Pivot.* Assuming the pivot to press uniformly on the bearing, the friction on an element of area $r'd\theta dr'$ is

$$f\frac{P}{\pi r^2}r'd\theta dr'. \quad \text{(Fig. 245.)}$$

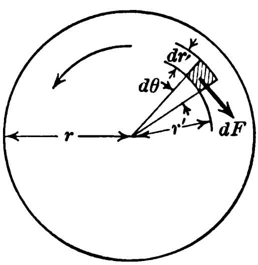


Fig. 245

The total work done against friction per revolution is

$$\text{Work per rev.} = \int_0^r \int_0^{2\pi} 2\pi r' f\frac{P}{\pi r^2}r'dr'd\theta$$

$$= \frac{2fP}{r^2}\int_0^r \int_0^{2\pi} r'^2 dr'd\theta,$$

or

$$\textbf{Work per revolution} = \frac{4\pi r f P}{3}.$$

(*b*) *Collar Bearing* or *Hollow Pivot.*

respectively, and T the tension in the belt at any point of the arc of contact. Consider the forces acting on a portion of the belt of length $\Delta s = r\Delta\beta$ (Fig. 242). These forces are T, $T+\Delta T$, and ΔF, tangent to the arc, and ΔP, the normal pressure of the pulley on the belt, which may be regarded as acting at the center of the arc. Let m be the mass of a unit length of the belt. The length of the portion considered is then $mr\Delta\beta$.

Suppose the belt to have uniform speed, v. The forces along the tangent will then balance, *i.e.*

$$T + \Delta F = T + \Delta T.$$
$$\therefore\ dF = dT.$$

The acceleration toward the center is $\frac{v^2}{r}$, and the force toward the center is

$$(T + \Delta T + T)\sin\frac{\Delta\beta}{2} - \Delta P.$$
$$\therefore\ (2\,T + \Delta T)\sin\frac{\Delta\beta}{2} - \Delta P = mv^2\Delta\beta.$$

Dividing by $\Delta\beta$ and passing to the limit, remembering that

$$\lim\left[\frac{\sin\frac{\Delta\beta}{2}}{\frac{\Delta\beta}{2}}\right] = 1, \text{ we have,}$$

$$T - \frac{dP}{d\beta} = mv^2.$$

If f is the coefficient of friction, and the belt is on the point of slipping,

$$dF = f dP,$$

or

$$dP = \frac{1}{f}dF = \frac{1}{f}dT.$$

Substituting this value of dP, we have

$$T - \frac{1}{f}\frac{dT}{d\beta} = mv^2.$$

$$\therefore \frac{dT}{T - mv^2} = f d\beta.$$

Integrating,

$$\log_e(T - mv^2)\Big|_{T_2}^{T_1} = f\beta\Big|_0^{\alpha},$$

or

$$\log_e \frac{T_1 - mv^2}{T_2 - mv_2} = f\alpha.$$

$$\therefore \frac{T_1 - mv^2}{T_2 - mv^2} = e^{f\alpha}.$$

For low velocities the term mv^2 is small compared to T_1 and T_2, and the formula may be written

$$\frac{T_1}{T_2} = e^{f\alpha}.$$

It should be noted that m in the above formula is the mass of a portion of the belt 1 foot long, and that v must be reckoned in feet per second if g is taken as 32.2.

Problem 387. A rope makes two complete turns around a post 6 in. in diameter. What maximum tension could be balanced by a force of 100 lb. if $f = .3$?

Problem 388. A weight of 500 lb. is to be lowered by a rope wound round a horizontal drum. If the arc of contact is 450° and the coefficient of friction is .25, what force is necessary to lower the weight uniformly?

Problem 389. The velocity of a belt is 3000 ft./min., the tension in the tight side is 150 lb. per inch of width, the coefficient of friction is .25, and the weight of a portion of the belt 1 ft. long and 1 in. wide is .15 lb. If the belt is on the point of slipping and the arc of contact is 150°, what is the tension in the slack side?

Problem 390. A rope is wrapped four times around a post and a man exerts a pull of 50 lb. on one end. If the coefficient of friction is .3, what force can be exerted upon a boat attached to the other end of the rope?

166. Power Transmitted by a Belt. — From the relation

$$dF = dT \qquad \text{(Art. 165.)}$$

there follows

$$\int_{T_2}^{T_1} dT = \int_0^F dF,$$

or

$$T_1 - T_2 = F, \text{ the total friction.}$$

The work done per second against F is

$$Fv \text{ or } (T_1 - T_2)v.$$

Hence the horse power transmitted by the belt is

$$\text{H.P.} = \frac{(T_1 - T_2)v}{550},$$

where T_1 and T_2 are in pounds and v is in feet per second.

Problem 391. Show that the formula for H.P. transmitted by the belt may be reduced to the form

$$\text{H.P.} = \frac{(1 - e^{-f\alpha})(T_1 - mv^2)v}{550}.$$

Problem 392. Given a maximum allowable tension, T_1, show that the power transmitted is a maximum when

$$v = \sqrt{\frac{T_1}{3m}}.$$

Problem 393. A pulley 4 ft. in diameter making 200 r.p.m. drives a belt that absorbs 20 H.P. The belt is $\frac{1}{4}$ in. thick and weighs 56 lb. per cubic foot. If $\alpha = \pi$ and $f = .27$, how wide must the belt be that the tension may not exceed 75 lb. per inch of width?

Problem 394. What H. P. may be transmitted by a belt 6 in. wide, $\frac{1}{4}$ in. thick, weighing 56 lb. per cubic foot, when traveling at 1500 ft. per minute, if $\alpha = 108°$, $f = .25$, and the maximum tension is 300 lb. per square inch?

Problem 395. Find the maximum H. P. that can be transmitted by the belt in the preceding problem, and the corresponding velocity.

167. Transmission Dynamometer. — It has been shown, in Art. 165, that the tension of a belt on the tight side is greater than the tension on the slack side. The transmission dynamometer (the Fronde dynamometer), illustrated in Fig. 243, is designed to measure the difference in these tensions. Let the pulley D be the driver and the pulley E the follower, so that T_1 represents the tight side of the belt and T_2 the slack side. The pulleys B, B run loose on the T-shaped frame CBB. This frame is pivoted at A. If we neglect the friction due to the loose pulleys, we have the following forces acting on the T-frame, two forces T_1 at the center of the right-hand pulley B, two forces T_2 at the center of the left-hand pulley B, a measurable reaction P at C, and the reaction of the pin at A. Taking moments about the pin, we have

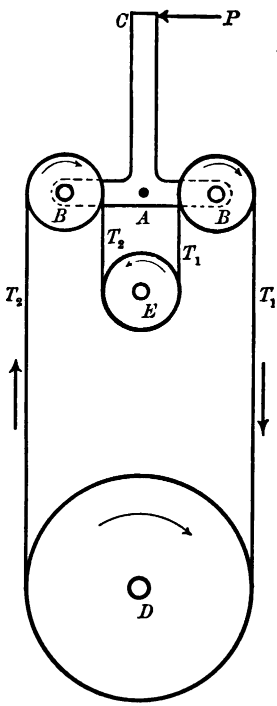


FIG. 243

nearly approached when the balls are hard and not easily changed from their spherical shape. All materials, however, are deformed under pressure so that perfect rolling friction is impossible. On account of the sliding friction present in roller and ball bearings, it is necessary to use a lubricant to prevent wear.

Problem 383. How many $\frac{3}{4}$-in. balls will be necessary in a ball bearing designed to carry 4000 lb., if $c = 7.5$? If $f = .0015$, what work is lost per revolution, the distance from the axis of rotation to the center of balls being one inch?

164. Friction Gears. — In the friction gears the driver is usually the smaller wheel, and when there is any difference in the materials of which the wheels are made, the driver is made of the softer material. This latter arrangement is resorted to, to prevent flat places being worn on either wheel in case of slipping. These gears have been used for transmitting light roads at high speeds, where toothed gears would be very noisy, or in cases where it is necessary to change the speed or direction of the motion quickly.

The use of paper drivers has made possible the transmission of much heavier loads by means of such gears. A series of tests, made by W. F. M. Goss, and reported in Trans. Am. Soc. M. E., Vol. 18, on the friction between paper drivers and cast-iron followers, is of interest in this connection. The apparatus used is shown in Fig. 240. The pressure between the wheels was obtained by a mechanism that forced the two wheels together with a pressure P. A brake wheel shown in the figure absorbed the power transmitted.

The coefficient of friction was regarded as the ratio of F to P, as in sliding friction. While this is customary, it is not entirely true, since we have the rolling of one

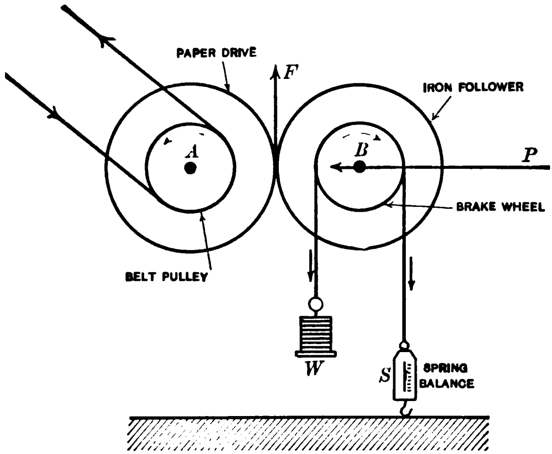


FIG. 240

body over the other. We shall, however, assume that we may call the coefficient of friction $f = \frac{F}{P}$. It was found that the coefficient of friction varied with the slippage, but was fairly constant for all pressures up to some point between 150 to 200 lb. per inch of width of wheel face. "*Variations in the peripheral speed between* 400 *and* 2800 *ft. per minute do not affect the coefficient of friction.*"

If the allowable coefficient of friction be taken as .20, the horse power transmitted per inch of width of face of the wheel, for a pressure of 150 lb., is

$$\text{H.P.} = \frac{150 \times .2 \times \frac{1}{12}\pi d \times w \times N}{33{,}000} = .000238\, dwN,$$

where d is the diameter of the friction wheel in inches, w the width of its face in inches, and N the number of revolutions per minute. Using this formula, the following table is given in the article in question:

HORSE POWER WHICH MAY BE TRANSMITTED BY MEANS OF PAPER FRICTION WHEEL OF ONE INCH FACE, WHEN RUN UNDER A PRESSURE OF 150 LB.

DIAMETER OF PULLEY IN INCHES	REVOLUTIONS PER MINUTE							
	25	50	75	100	150	200	600	1000
8	.0476	.0952	.1428	.1904	.2856	.3808	1.1424	1.904
10	.0595	.1190	.1785	.2380	.3570	.4760	1.4280	2.380
14	.0833	.1666	.2499	.3332	.4998	.6664	1.9992	3.332
16	.0952	.1904	.2856	.3808	.5712	.7616	2.2848	3.808
18	.1071	.2142	.3213	.4284	.6426	.8568	2.5704	4.288
24	.1428	.2856	.4284	.5712	.8568	1.1424	3.4272	5.712
30	.1785	.3570	.5355	.7140	1.0710	1.4280	4.2840	7.140
36	.2142	.4284	.6426	.8568	1.2852	1.7136	5.1408	8.560
42	.2499	.4998	.7497	.9996	1.4994	1.9992	5.9976	9.996
48	.2856	.5712	.8568	1.1424	1.7136	2.2848	6.8544	11.420

The value of the coefficient of friction for friction gears (Kent, "Pocket-Book") may be taken from .15 to .20 for metal on metal; .25 to .30 for wood on metal; .20 for wood on compressed paper.

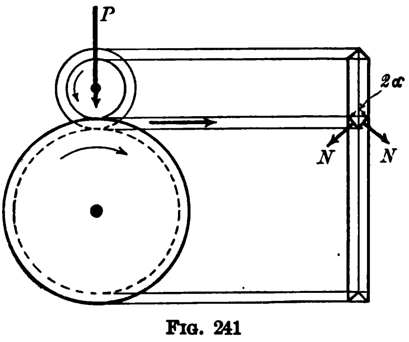


FIG. 241

Problem 384. If the friction wheels are grooved as shown in Fig. 241, both

of cast iron, and the small driver fits into the groove of the larger follower, prove that the force transmitted is

$$F = 2fN = \frac{fP}{\sin\alpha}.$$

Problem 385. The speed of the rim of two grooved friction wheels is 400 ft. per minute. If $\alpha = 45°$, $f = .18$, what must be the pressure P to transmit 100 horse power?

Problem 386. What horse power may be transmitted by the gearing in the preceding problem, if $P = 6000$ lb. and the peripheral velocity is 12 ft. per second?

165. Friction of Belts. — When a belt or cord passes over a pulley and is acted upon by tensions T_1 and T_2, the tensions are unequal, due to the friction of the pulley on the belt. We shall determine the relation between T_1 and T_2. Let the pulley be represented in Fig. 242. The belt

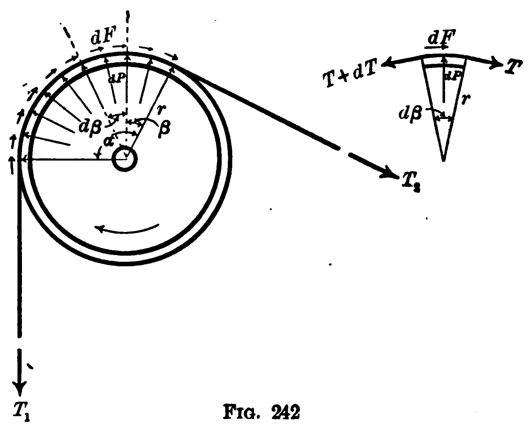


FIG. 242

covers an arc of the pulley whose angle is α. Consider the forces acting upon the belt and suppose T_1 and T_2 to be the tensions in the belt on the tight and slack sides,

Let dM be any element of mass of the body, distant r from the y-axis. The velocity of dM is then composed of the velocity of any point on the y-axis and the velocity of dM relative to the y-axis. Hence, v, the velocity of dM, is given by

$$v^2 = v_1^2 + r^2\omega^2 + 2\,v_1 r\omega\cos\phi,$$

where ϕ is the angle which r makes with a line parallel to the z-axis. But $r\cos\phi = z$, so that

$$v^2 = v_1^2 + r^2\omega^2 + 2\,v_1\omega z.$$

The kinetic energy of the whole body is the sum of the kinetic energy of its particles, or

$$\begin{aligned}\text{K. E. of Body} &= \int \tfrac{1}{2}(v_1^2 + r^2\omega^2 + 2\,v_1\omega z)dM \\ &= \tfrac{1}{2}\,v_1^2\int dM + \tfrac{1}{2}\,\omega^2\int r^2 dM + 2\,v_1\omega\int z dM.\end{aligned}$$

Since the xy-plane passes through the center of gravity,

$$\int z dM = 0. \qquad \text{(Art. 33.)}$$

Therefore

$$\textbf{K. E. of Body} = \tfrac{1}{2}\,Mv_1^2 + \tfrac{1}{2}\,\mathbf{I}\omega^2,$$

where M is the mass of the body and I its moment of inertia about a gravity axis perpendicular to the plane of motion.

This formula may be expressed in words as follows: *The kinetic energy of a body having plane motion is equal to the kinetic energy the whole mass would have if concentrated at the center of gravity, with the velocity of the center of gravity, plus the kinetic energy of rotation that the body*

would have if the gravity axis were at rest and the body rotating about it with the same angular velocity.

150. Work and Kinetic Energy in Plane Motion. — If impressed forces act on a body having plane motion, the work done by these forces in any displacement equals the change in kinetic energy of the body in that displacement.

Proof: The total gain in the kinetic energy of the body is the sum of the increments in kinetic energy of all the particles of the body, which, by Art. 143, is the total work done on the particles by all the forces acting on them. In finding the total work done on the particles the work done by the forces that the particles exert on each other adds up to zero and there is left only the work done by the impressed forces. For two particles of the body, m_1 and m_2, remain the same distance apart and exert on each other forces that are equal and opposite and in the line joining the particles. During any displacement m_2 has a motion made up of the motion of m_1 and a motion about m_1 as a center (Fig. 221).

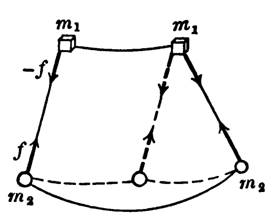


FIG. 221

For that component of its motion parallel to the motion of m_1 the forces exerted by the particles on each other are equal and opposite and have the same displacement, and hence the work done by one force is just equal to the work done against the other. For the motion of rotation of m_2 about m_1 the force acting on m_2 is perpen-

dicular to the direction of displacement, and hence no work is done. Hence the forces exerted by the particles on each other do no work, and we may write for any plane motion,

Work done by impressed forces = change in kinetic energy of body.

As an illustration, consider a body of circular section, as a hoop, cylinder, or sphere, with center of gravity at the center of the circular section, rolling without slipping down an inclined plane (Fig. 222). The impressed forces acting on the body are its weight, G, the normal reaction of the plane, N, and a retarding friction force, F, along the plane. The point of application of N has no displacement in the direction in which N acts, and hence no work is done by N. The point of application of F, continually changing in the body, has at each instant a motion at right angles to F; for if there is no slipping, the point of the body in contact with the plane at any instant leaves the plane at right angles to the plane. Hence no work is done by F. The total work done in the descent is therefore Gh.

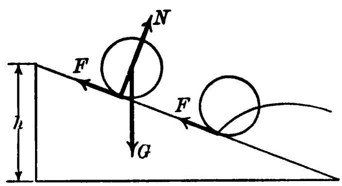


FIG. 222

If ω_0 and v_0 are the angular velocity of the body and the linear velocity of its center of gravity respectively at the top of the plane and ω and v the corresponding values at the foot, then the work-energy equation is

$$\tfrac{1}{2} I\omega^2 + \tfrac{1}{2} Mv^2 - \tfrac{1}{2} I\omega_0^2 - \tfrac{1}{2} Mv_0^2 = Gh.$$

If s is the distance passed through by the center and θ

the angle turned through in the same time,

$$s = r\theta.$$

Therefore $$\frac{ds}{dt} = r\frac{d\theta}{dt},$$

or $$v = r\omega.$$

Substituting $$\omega = \frac{v}{r},\ \omega_0 = \frac{v_0}{r},\ I = \frac{G}{g}k^2,\ M = \frac{G}{g},$$

the work-energy equation becomes

$$(v^2 - v_0^2)\left(\frac{k^2}{r^2} + 1\right) = 2gh,$$

or $$v^2 - v_0^2 = \frac{2gh}{1 + \frac{k^2}{r^2}}.$$

Problem 362. Prove that all solid spheres will roll down the inclined plane at the same rate. Find the velocity at the foot of the plane.

Problem 363. A uniform sphere, a uniform disk, and a hoop, starting at the top of an inclined plane, roll from rest to the foot. Find the velocity of each on reaching the foot. In what order do they arrive?

Problem 364. Which will roll faster down an inclined plane, a hollow sphere with diameter of the hollow one half that of the sphere, or a solid uniform disk?

151. Kinetic Energy of Rolling Bodies. — It is convenient to express the kinetic energy or combined rotation and translation of such bodies as rolling wheels in a different form from that given in the preceding article. There is some mass M_1 that will have the same kinetic energy when translated with a velocity v_1 as the kinetic energy

of translation plus the kinetic energy of rotation of the body of mass M; that is,

$$\frac{M_1 v_1^2}{2} = \frac{M v_1^2}{2} + \frac{\omega^2 I}{2}.$$

For a wheel rolling on a straight track $\omega r = v_1$, where r is the radius.

Then $$M_1 = M + \frac{I}{r^2}.$$

This has been called the equivalent mass.

For example, for a rolling disk, since $I = \frac{1}{2} Mr^2$, $M_1 = \frac{3}{2} M$.

Problem 365. A sphere rolls without slipping down an inclined plane. Show that its kinetic energy is the same at any instant as that of a sphere whose mass is $\frac{2}{5}$ larger translated with a velocity equal to the velocity of the center of gravity of the rolling sphere.

Problem 366. The sphere in the preceding problem is made of steel, 12 in. in diameter, and the inclination of the plane is 30°. If $v_0 = 10$ ft. per second, what will be the velocity 10 ft. down the plane?

152. Work-energy Relation for Any Motion. — The relation between work and energy for the motions considered in this chapter holds for more complicated motions and for motions in general. The limits of the present work will not admit the proof of the general theorem. It may be said, however, that for any motion the work done by the working forces equals the work done by the resisting forces plus the change in kinetic energy. In the case of the motion of a complicated machine, the work done by the working forces equals the work done against the resistances plus the gain in kinetic energy of the various parts of the machine.

153. Work Done when Motion is Uniform. — When the motion is uniform, the change in kinetic energy is zero, and the work-energy equation reduces to the form: *work done equals the work done against the resistance overcome.*

As an illustration, let us consider the case of a locomotive moving at uniform speed and represented in Fig. 223. Suppose P the mean effective steam pressure (See Art. 139), F the friction of the piston, F' the friction of the crosshead, F'' the journal friction, F''' the crank-pin

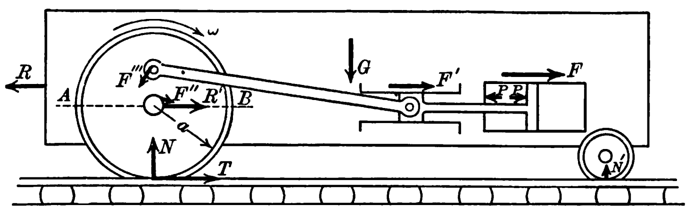


FIG. 223

friction, T the friction on the rail, R the draw-bar resistance, R' the horizontal component of pressure of the drivers' axle on the frame, r, r_1, and r_2 the radii of the crank-pin circle, the driver axle, and the crank-pin respectively, G the weight of the locomotive, and N' and N the normal reactions of the rails on the wheels. Consider both sides of the locomotive and write the work-energy equation for a distance s, equal to a half turn of the driver (from dead center A to dead center B), that the locomotive travels.

Considering the work done on the frame, counting both cylinders, we have

$$R'\pi a + 2(F + F')\pi a + 2F''\pi a = (2P + R)\pi a. \quad (1)$$

Considering the work done on the rotating and oscillating parts,

$$2P(\pi a + 2r) = R'\pi a + 2(F + F')(\pi a + 2r) + 2F''(\pi a + \pi r_1) + 2F'''\pi r_2. \quad (2)$$

Adding these equations there results

$$4Pr = R\pi a + (F + F')4r + 2F''\pi r_1 + 2F'''\pi r_2. \quad (3)$$

If we neglect friction, this equation becomes

$$4Pr = \pi a R,$$

or

$$P = \frac{\pi a}{4r}R.$$

Here P is the mean effective pressure in one cylinder.

This is the formula usually given for the tractive power of a locomotive having single expansion engines. This may be expressed in terms of the dimensions of the cylinders and the unit steam pressure. Let p be the unit steam pressure in pounds per square inch, l the length of the cylinder in inches, d the diameter of the cylinder in inches, and d_1 the diameters of the drivers in inches; then

$$R = \frac{d^2pl}{d_1}.$$

For uniform motion the train resistance cannot exceed the friction force or force of adhesion between the drivers and the rails, since these are the external horizontal forces acting on the engine at any time. This force of adhesion in American practice is usually taken as $\frac{1}{4}$ or $\frac{1}{5}$ of the weight on the drivers.

Problem 367. Derive equation (3) of this article by considering the work done on the whole engine.

Problem 368. What resistance R may be overcome by a locomotive moving at uniform speed, diameter of drivers 62 in., cylinders 16×24 in., and a steam pressure on the piston of 160 lb. per square inch? What should be the weight of the locomotive on the drivers?

Problem 369. If the diameter of the drivers of a locomotive is 68 in., and the size of the cylinder is 20×24 in., what train resistance may be overcome by a steam pressure of 160 lb. per square inch?

Problem 370. A locomotive has a weight of 155 tons on the drivers. If the adhesion is taken as $\frac{1}{5}$, this allows 31 tons for the drawbar pull. The train resistance per ton of 2000 lb., for a speed of 60 mi. per hour, is 20 lb. Find the weight of the train that can be pulled by the locomotive at the speed of 60 mi. per hour.

Problem 371. An 80-car freight train is to be pulled by a single expansion locomotive at the rate of 30 mi. per hour. The weight of each car is 60,000 lb., and the resistance for this speed is 10 lb. per ton. What must be the weight on the drivers, if the adhesion is $\frac{1}{5}$?

CHAPTER XIII

FRICTION

154. Friction. — When one body is made to slide over another, there is considerable resistance offered because of the roughness of the two bodies. A book drawn across the top of a table is resisted by the roughness of the two bodies. The rough parts of the book sink into the rough parts of the table so that when one of the bodies tends to move over the other, the projections interfere and tend to stop the motion. The bearings of machines are made very smooth, and usually we do not think of such surfaces as having projections. Nevertheless they are not perfectly smooth, and when one surface is rubbed over the other, resistance must be overcome. This resisting force to the motion of one body over another is known as *friction*. When the bodies are at rest relative to each other, the friction is known as the *friction of rest*, or *static friction*. When they are in motion with respect to each other, the friction is known as the *friction of motion*, or *kinetic friction*.

155. Coefficient of Friction. — If the body represented in Fig. 224 be pulled along the horizontal plane by the force P, the following forces will be acting on it: the downward force G and the reaction R inclined back of the vertical through the angle θ. The reaction R of the plane on the body may be resolved into two components, one horizontal

and one vertical. The horizontal force is known as the force of friction, and the normal force, the normal pressure.

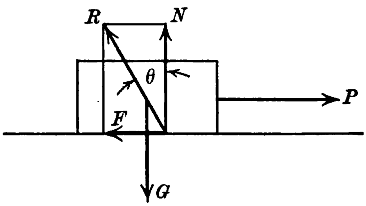


FIG. 224

The tangent of the angle θ, or $\frac{F}{N}$, is called the *coefficient of friction.* This coefficient of friction, which we shall represent by f, may be defined as *the ratio of the force of friction to the normal pressure; it is an absolute number.*

The angle θ, between R and the normal to the surface of contact, is called the *angle of friction.*

$$\theta = \tan^{-1} f.$$

The coefficient of friction is usually determined by allowing a body to slide down an inclined plane, as shown in Fig. 225. The angle θ is increased until the force of friction F will just keep the body from sliding down the plane. The angle θ is then called the *angle of repose*, and the tangent of θ is the coefficient of friction.

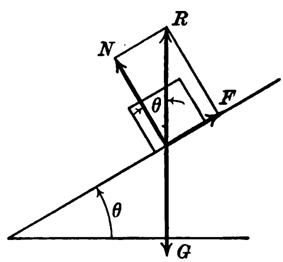


FIG. 225

Proof: When the body is just on the point of slipping down, the force R must just balance G. Hence the angle between R and N is equal to the angle of inclination of the plane, or the angle of friction is equal to the angle of inclination of the plane.

It is possible with such an apparatus to determine the coefficient of friction for various materials. It has been found that after motion begins the friction is less; that is, *the friction of motion is less than the friction of rest.* This is an important law for engineers.

156. Laws of Friction for Dry Surfaces. — Very little was known of the laws of friction until within the last seventy-five years. About 1820 experiments were made that seemed to show that, for such materials as wood, metals, etc., friction varies with the pressure, and is independent of the extent of the rubbing surfaces, the time of contact, and the velocity. A little later (1831) Morin published the following three laws as a result of his experiments on friction:

(1) *The friction between two bodies is directly proportional to the pressure; that is, the coefficient of friction is constant for all pressures.*

(2) *The coefficient and amount of friction for any given pressure is independent of the area of contact.*

(3) *The coefficient of friction is independent of the velocity, although static friction is greater than kinetic friction.*

These laws of Morin hold approximately for dry, unlubricated surfaces, although it has been found that an increase in speed lowers the coefficient of friction. The coefficient of friction is a little greater for light pressures upon large areas than for great pressures on small areas.

The following is a table of some of the coefficients of friction as determined by Morin :

COEFFICIENTS OF FRICTION, DUE TO MORIN

MATERIAL	CONDITION OF SURFACE	COEFFICIENT OF FRICTION	ANGLE OF FRICTION
Brick on limestone	Dry	.67	33° 50′
Cast iron on cast iron	Slightly greased	.16	9° 6′
Cast iron on oak	Wet	.65	33° 2′
Copper on oak		.17	9° 38′
Copper on oak	Greased	.11	6° 17′
Leather on cast iron		.28	15° 39′
Leather on cast iron	Wet	.38	20° 49′
Leather on cast iron	Oiled	.12	6° 51′
Leather on oak	Fibers parallel	.74	36° 30′
Leather on oak	Fibers crossed	.47	25° 11′
Oak on oak	Fibers parallel, dry	.62	31° 48′
Oak on oak	Fibers crossed, dry	.54	28° 22′
Oak on oak	Fibers parallel, soaped	.44	23° 45′
Oak on oak	Fibers crossed, wet	.71	35° 23′
Oak on oak	Fibers end to side, dry	.43	23° 16′
Oak on oak	Fibers parallel, greased	.07	4° 6′
Oak on oak	Heavily loaded, greased	.15	8° 45′
Oak on pine	Fibers parallel	.67	33° 50′
Oak on limestone	Fibers on end	.63	32° 15′
Oak on hemp cord	Fibers parallel	.80	38° 40′
Pine on pine	Fibers parallel	.56	29° 15′
Pine on oak	Fibers parallel	.53	27° 56′
Wrought iron on oak	Wet	.62	31° 48′
Wrought iron on oak		.65	33° 2′
Wrought iron on wrought iron		.28	15° 39′
Wrought iron on cast iron		.19	10° 46′
Wrought iron on limestone		.49	26° 7′
Wood on metal	Greased	.10	6° 0′
Wood on smooth stone	Dry	.58	30° 7′
Wood on smooth earth	Dry	.33	18° 16′

Problem 372. Find the force P necessary to move with uniform velocity a weight of 100 lb. up a plane inclined 30° to the horizontal (*a*) when P is horizontal, (*b*) when parallel to the plane, (*c*) when inclined at an angle of 60° to the horizontal, given that the coefficient of friction between the weight and the plane is .20. Find the force just necessary to prevent the body from sliding down the plane in each of the cases.

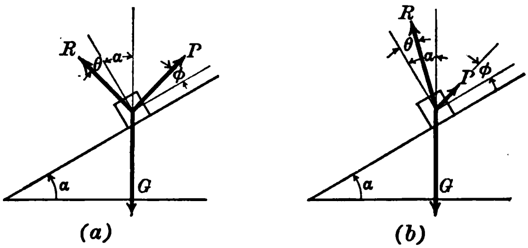


FIG. 226

Problem 373. Show that the force P, inclined at an angle ϕ to the plane, that will (*a*) just move the weight up the plane, (*b*) just prevent it from sliding down the plane (Fig. 226), is

$$(a)\ P = \frac{\sin(\alpha + \theta)}{\cos(\phi - \theta)} G, \qquad (b)\ P = \frac{\sin(\alpha - \theta)}{\cos(\phi + \theta)} G,$$

where θ is the angle of friction.

Problem 374. Show that the least values of P in the preceding problem are when $\phi = \theta$ in (*a*), and when $\phi = -\theta$ in (*b*); *i.e.* when

$$(a)\ P = G \sin(\alpha + \theta), \qquad (b)\ P = G \sin(\alpha - \theta).$$

Problem 375. In Fig. 227 the weight G is raised by the horizontal force P. If the only friction is between the surfaces of the wedge and the weight, prove that the value of P just sufficient to raise the weight is

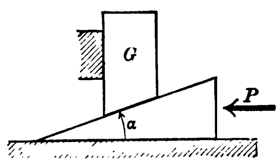


FIG. 227

$$P = G \tan(\alpha + \theta).$$

Problem 376. Defining the efficiency, E, of the wedge as the ratio of the useful work accomplished in raising the weight to the total work done by P (Fig. 227) show that

$$E = \frac{\tan \alpha}{\tan (\alpha + \theta)}.$$

Problem 377. For a given value of θ, show that E is a maximum when

$$\alpha = 45° - \frac{\theta}{2}.$$

(Since a square-threaded screw may be regarded as an inclined plane, this formula also holds for such a screw.)

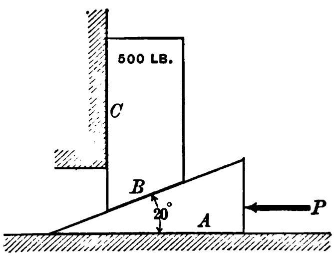


FIG. 228

Problem 378. Find the value of the horizontal force P that will just raise the weight of 500 lb. in Fig. 228, given that the coefficients of friction at A, B, and C are respectively .20, .25, and .30.

SUGGESTION. Consider the wedge and block separately, and the forces that hold them in equilibrium.

157. Friction of Lubricated Surfaces. — The laws of friction, as given by Morin and stated in the preceding article, hold approximately for rubbing surfaces, when the surfaces are dry or nearly so; that is, for poorly lubricated surfaces. If, however, the surfaces are well lubricated so that the projections of one do not fit into the other, but are kept apart by a film or layer of the lubricant, the laws of Morin are not even approximately true. The study of the friction of lubricated surfaces, then, may be divided into two parts: (1) the study of poorly lubricated bearings, and (2) the study of well lubricated bearings, the

friction of which varies from $\frac{1}{6}$ to $\frac{1}{10}$ that of dry or poorly lubricated bearings.

Since the friction of poorly lubricated bearings is about the same as that of dry surfaces, we shall consider that the laws of Morin hold, and shall confine our attention to the friction of well lubricated bearings. If the lubricant is an oil, the friction of the bearing is no longer due to one surface rubbing over the other, but to the friction between the bearing and the oil, and to the internal friction of the oil. That is, the oil adheres to the two surfaces, and its own particles attract each other, and the motion of one of the surfaces with respect to the other changes the positions of the oil particles. It is to be expected, then, that the friction of an oiled bearing will depend upon the *viscosity of the oil*, upon the *thickness of the layer interposed between the surfaces*, and upon the *velocity and form of the bearing.*

The coefficient of friction is no longer constant, but varies with the temperature, velocity, and pressure. The variation of the coefficient of friction of a paraffine oil with temperature is shown in Fig. 229 when the pressure on the bearing is 33 lb. per square inch and a velocity of rubbing of 296 ft. per minute. It is seen that the coefficient of friction *decreases* with increase of temperature until a temperature of 80° F. is reached, when it increases rapidly. This means that above this temperature the oil is so thin that it is squeezed out of the bearing, and the conditions of dry bearing are approached. The temperature at which oils show an increasing coefficient of friction is different for different oils, even at the same pressure and

velocity. The curve in Fig. 229, however, may be regarded as typical of all oils when the pressure and velocity are constant.

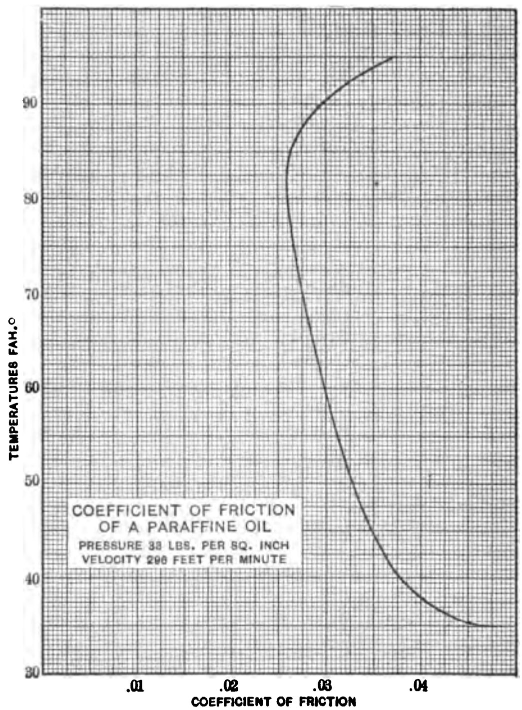


FIG. 229

The following table, due to Professor Thurston, shows the relation between the coefficient of friction and temperature for a sperm oil in steel bearings when the velocity of rubbing is 30 ft. per minute:

Pressure, Lb. per Sq. In.	Temperature, Degrees F.	Coefficient of Friction	Pressure, Lb. per Sq. In.	Temperature, Degrees F.	Coefficient of Friction
200	150	.0500	100	110	.0025
200	140	.0250	50	110	.0035
200	130	.0160	4	110	.0500
200	120	.0110	200	90	.0040
200	110	.0100	150	90	.0025
200	100	.0075	100	90	.0025
200	95	.0060	50	90	.0035
200	90	.0056	4	90	.0400
150	110	.0035			

It is seen that for a pressure of 200 lb. per square inch as the temperature increases from 90° F. the coefficient increases, indicating that the temperature of 90°, for the given pressure and velocity, was above the temperature at which the oil became so thin as to be squeezed out and the bearing to approach the condition of a dry bearing. For a constant temperature 110° F. and 90° F. the coefficient is seen to decrease with increase of pressure up to a certain point and then to increase. This is a typical behavior of oils when the temperature is constant and the pressure varies.

At speeds exceeding 100 ft. per minute, the same authority found "that the heating of the bearings within the above range of temperatures decreases the resistance due to friction, rapidly at first and then slowly, and gradually a temperature is reached at which increase takes place and progresses at a rapidly accelerating rate."

The relation between the coefficients of rest and of motion as determined by Professor Thurston for three oils is given

below. The journals were cast iron, in steel boxes ; velocity of rubbing 150 ft. per minute and a temperature 115° F.

	Sperm Oil			West Virginia Oil			Lard		
Pressure, Lb. per Sq. In.	At 150 ft. per min.	At starting	At stopping	At 150 ft. per min.	At starting	At stopping	At 150 ft. per min.	At starting	At stopping
50	.013	.07	.03	.0213	.11	.025	.02	.07	.01
100	.008	.135	.025	.015	.135	.025	.0137	.11	.0225
250	.005	.14	.04	.009	.14	.026	.0085	.11	.016
500	.004	.15	.03	.00515	.15	.018	.00525	.10	.016
750	.0043	.185	.03	.005	.185	.0147	.0066	.12	.020
1000	.009	.18	.03	.010	.18	.017	.0125	.12	.019
	Steel Journals and Brass Boxes								
500	.0025						.004		
1000	.008						.009		

It is seen that the coefficient of friction at starting is much greater than at stopping, and that these are both much greater than the value at a speed of 150 ft. per minute.

For an intermittent feed such as is given by one oil hole, without a cup, oiled occasionally, Professor Thurston found for steel shaft in bronze bearings, with a speed of rubbing of 720 ft. per minute, the following coefficients of friction :

Oil	Pressure, Lb. per Sq. In.			
	8	16	32	48
Sperm and lard	.159–.25	.138–.192	.086–.141	.077–.144
Olive and cotton seed .	.160–.283	.107–.245	.101–.168	.079–.131
Mineral oils	.154–.261	.145–.233	.086–.178	.094–.222

The results show that the coefficient decreases with the pressure within the range reported, but that the results are considerably higher than those for well lubricated bearings. He also found in connection with the same tests that with *continuous lubrication* sperm oil gave the following coefficients:

Pressure, Lb. per Sq. In.	Coefficient of Friction
50	.0034
200	.0051
300	.0057

The results of tests of the friction of well-lubricated bearings are summarized by Goodman (*Engineering News*, April 7 and 14, 1888) as follows:

(*a*) *The coefficient of friction of well lubricated surfaces is from* $\frac{1}{6}$ *to* $\frac{1}{10}$ *that of dry or poorly lubricated surfaces.*

(*b*) *The coefficient of friction for moderate pressures and speeds varies approximately inversely as the normal pressure; the frictional resistance varies as the area in contact, the normal pressure remaining the same.*

(*c*) *For low speeds the coefficient of friction is abnormally high, but as the speed of rubbing increases from about* 10 *to* 100 *ft. per minute, the coefficient of friction diminishes, and again rises when that speed is exceeded, varying approximately as the square root of the speed.*

(*d*) *The coefficient of friction varies approximately inversely as the temperature, within certain limits; namely, just before abrasion takes place.*

158. Method of Testing Lubricants. — To make the matter of the tests of the friction of lubricants clear, it will be

convenient to make use of the description of a testing machine used by Dean W. F. M. Goss at Purdue University on graphite, and a mixture of graphite and sperm

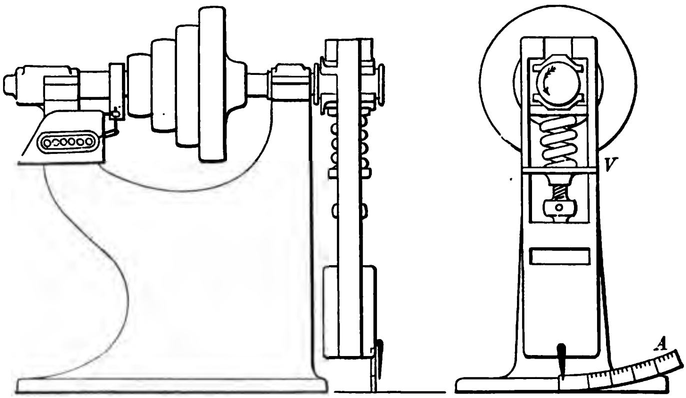


FIG. 230

oil. In making the tests the apparatus shown in Figs. 230 and 231 was used. (See "A Study in Graphite," Joseph Dixon Crucible Co.)

This apparatus represents, in principle, the machines generally used for testing lubricants. It is therefore shown in some detail. The weight G is hung from the shaft upon which it is suspended by the form of box to be tested. The desired speed of rubbing is obtained by means of the cone of pulleys, and the pressure on the bearing is adjusted by the spring. The temperature of the bearing is read from the thermometer inserted in the bearing. When rotation takes place, the weight G is rotated a certain distance dependent upon the friction. This distance is measured on the scale A. The forces acting upon

the pendulum G are shown in Fig. 231, where R represents the resistance of the spring, F the force of friction, l the distance of the center of gravity of G from the axis of rotation, ϕ the angle through which G is deflected, r the radius of the shaft, and f the coefficient of friction. Taking moments about the center of the shaft, we have, when G is held in the position shown, due to the friction,

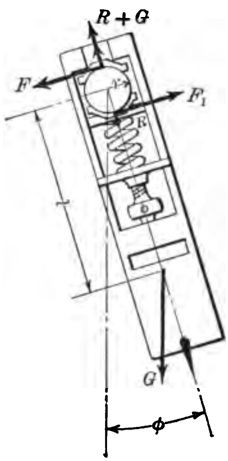


FIG. 231

$$r(F + F_1) = Gl \sin \phi,$$

or $$rf(R + G + R) = Gl \sin \phi.$$

$$\therefore f = \frac{Gl \sin \phi}{r(2R + G)}.$$

It is customary to take G small compared with R, so that the pressure on both sides of the bearing may be considered equal to R, the resistance of the spring. The formula then becomes

$$f = \frac{Gl \sin \phi}{2rR}.$$

The spring is easily calibrated so that R may be made anything desired by compressing the spring through the appropriate distance, as indicated on the scale V (Fig. 230). The quantities G, l, r, and R are known, and ϕ can be read so that f can be calculated.

The results of tests made upon a mixture of graphite and oil as a lubricant are given in the pamphlet. The tests were run under 200 lb. per square inch pressure, at a speed of rubbing of 145 ft. per minute. Oil was dropped

into the bearing at the rate of about 12 drops per minute, showing a coefficient of friction of $\frac{1}{4}$.

Problem 379. If the weight of the pendulum is 360 lb., the diameter of the shaft $4\frac{1}{2}$ in., distance of the center of gravity of G from the center of shaft 2 ft., the angle ϕ 5 degrees, and the average resistance of the spring 1000 lb., find the coefficient of friction. The weight G should not be neglected in this case.

159. Rolling Friction. — The resistance offered to the rolling of one body over another is known as rolling friction. It is entirely different from sliding friction, and its laws are not so well understood. When a wheel or cylinder (Fig. 232) rolls over a track, the track is depressed and the wheel distorted. The force P necessary to overcome this depression and distortion is known as *rolling friction.*

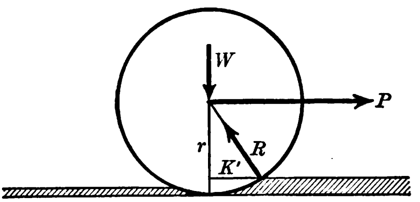


FIG. 232

The forces acting on the wheel are seen from Fig. 232 to be: P the working force, W the weight on the wheel, and R the reaction of the track or roadway. This upward pressure R is not quite vertical, but has its point of application a short distance K' from the vertical. Its line of action passes through the center of the wheel. The distance K' depends chiefly upon the roadway; it is called the *coefficient of rolling friction.* It is measured in inches and is not a coefficient of friction in the strict sense that f is the coefficient of sliding friction.

Taking moments about the point of application of R,

we have, approximately,

$$WK' = Pr,$$

so that $$K' = \frac{Pr}{W}, \text{ or } P = \frac{K'W}{r}.$$

When the track or roadway is elastic or nearly so, we have a condition something like that represented in Fig. 233. The wheel sinks into the material and pushes it ahead, at the same time it comes up behind the wheel. For a portion of the wheel on each side of the point O the roadway is simply compressed; over the remainder of the surface in contact, however, slipping occurs, as indicated by the arrows. The resultant resistance, however, is in front of the vertical through the center, and we have, as in the case of imperfectly elastic roadways,

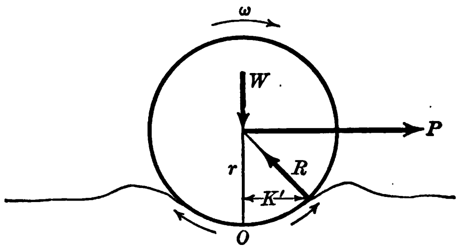


Fig. 233

$$P = \frac{K'W}{r}.$$

It has been found by Reynolds (See Phil. Trans. Royal Soc., Vol. 166, Part 1) that when a cast-iron roller rolls on a rubber track, the slippage, due to the elasticity of the track, may amount to as much as .84 in. in 34 in. An elastic roller rolling on a hard track will roll less than the geometrical distance traveled by a point on the circumference. When the roller and track are of the same material, the roller rolls through less than its geometrical distance.

160. Antifriction Wheels. — The axle A, of radius r, carrying a weight W, rests upon two wheels of radius r_2, turning on axles of radius r_1 (Fig. 234).

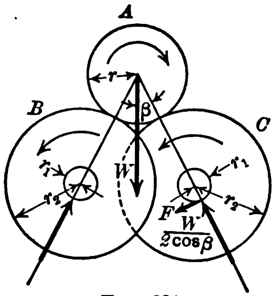


FIG. 234

The force on each of the bearings of wheels B and C is

$$\frac{W}{2\cos\beta},$$

and if F is the friction at each of the bearings of B and C, and f the coefficient of sliding friction,

$$F = f\frac{W}{2\cos\beta},$$

and the work lost in friction at the bearings of B and C in one revolution of the axle A is

$$f\frac{W}{\cos\beta}r_1\frac{2\pi r}{r_2}.$$

Since the axle A rolls on B and C, there is no sliding friction there and the rolling friction is small enough to be neglected.

If A were in an ordinary bearing, the work lost per revolution would be

$$f\,2\pi r W.$$

Therefore the ratio of work lost with antifriction wheels to the work lost with plain bearing is

$$\frac{r_1}{r_2\cos\beta}.$$

This ratio decreases as the ratio r_2/r_1 increases and as β decreases.

Problem 380. If $W = 4$ tons, the radius of the shaft is 2 in., and the coefficient of friction is .07, what work is lost per revolution? If the shaft makes 3 revolutions per second, what horse power is lost in friction? Given also $\beta = 45°$, $r_1 = \frac{3}{4}$ in., and $r_2 = 4$ in.

Problem 381. In the case of the shaft mentioned in the preceding problem, how much more horse power would it take if the bearing were plain? What value of β would give the same loss due to friction in both the plain bearing and the one provided with friction wheels?

161. Resistance of Ordinary Roads. — Resistance to traction consists of axle friction, rolling friction, and grade resistance. Axle friction varies from .012 to .02 of the load, for good lubrication, according to Baker. The tractive power necessary to overcome axle friction for ordinary American carriages has been found to be from 3 lb. to $3\frac{1}{2}$ lb. per ton, and for wagons with medium-sized wheels and axles from $3\frac{1}{2}$ lb. to $4\frac{1}{2}$ lb. per ton.

The total tractive force per ton of load, for wheels 50 in., 30 in., and 26 in., in diameter, respectively, is, according to Baker (*Engineering News*, March 6, 1902):

	Tractive Force in Pounds		
On macadam roads	57	61	70
On timothy and blue grass sod, dry, grass cut	132	145	179
On timothy and blue grass sod, wet and springy	173	203	288
On plowed ground, not harrowed, dry and cloddy	252	303	374

Rolling resistance is influenced by the width of the tire. According to Baker, poor macadam, poor gravel, compressible earth roads, and, on agricultural lands, narrow tires,

usually require less traction. On earth roads composed of dry loam with 2 to 3 in. of loose dust, traction with $1\frac{1}{2}$-in. tires was 90 lb. per ton, and with 6-in. tires 106 lb. per ton. On the same road when it was hard and dry, with no dust, that is, when it was compressible, the traction was found to be 149 lb. per ton with $1\frac{1}{2}$-in. tires and 109 lb. per ton with 6-in. tires. On broken stone roads, hard and smooth, with no dust or loose stones, the traction per ton was 121 lb. with $1\frac{1}{2}$-in. tires, and 98 lb. with 6-in. tires. Moisture on the surface or mud increases the traction.

Morin found that with 44-in. front and 54-in. rear wheels on hard dry roads the traction per ton was 114 lb. with either $1\frac{3}{4}$-in. or 3-in. tires. On wood-block pavements the traction per ton was 28 lb. with $1\frac{1}{2}$-in. tires, and 38 lb. with 6-in. tires.

On asphalt, bricks, granite, macadam, and steel-road surfaces, investigated by Baker, the traction per ton varied from 17 lb. to 70 lb., the average being 38 lb.

Morin gives the coefficient of rolling friction for wagons on *soft soil* as **.065** in., and on *hard roads* **.02** in. According to Kent ("Pocket-Book"), tests made upon a loaded omnibus gave the following results:

Pavement	Speed, Miles per Hour	Coefficient, Inches	Resistance, per Ton, in Lb.
Granite	2.87	.007	17.41
Asphalt	3.56	.0121	27.14
Wood	3.34	.0185	41.60
Macadam, graveled	3.45	.0199	44.48
Macadam, granite, new	3.51	.0451	101.09

Problem 382. Compare the resistance offered to a load of two tons pulled over asphalt, macadam, good earth roads, or wood-block pavement. Width of tires, 6 in.

162. Roller Bearings. — In the roller bearings the shaft rolls on hardened steel rollers as shown in cross section in Fig. 235. The rollers are kept in place in some way similar to that shown in the journal of Fig. 236. Such bearings are used where heavy loads are to be carried. Tests of roller bearings have been made by Dean C. H. Benjamin (*Machinery*, October, 1905), who determined the following values for the coefficient of friction. Speed 480 revolutions per minute.

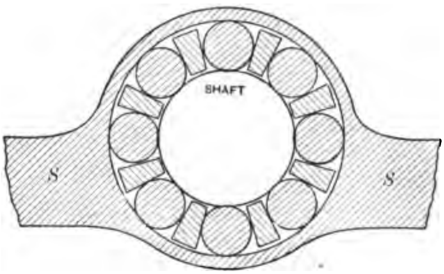


FIG. 235

Diameter of Journal, in Inches	Roller Bearing			Plain Cast-iron Bearing		
	Max.	Min.	Average	Max.	Min.	Average
$1\frac{15}{16}$	.036	.019	.026	.160	.099	.117
$2\frac{3}{16}$	.052	.034	.040	.129	.071	.094
$2\frac{7}{16}$	.041	.025	.030	.143	.076	.104
$2\frac{15}{16}$	.053	.049	.051	.138	.091	.104

It was found that the coefficient of friction of roller bearings is from $\frac{1}{5}$ to $\frac{1}{3}$ that of plain bearings at moderate speeds and loads. As the load on the bearing increased,

the coefficient of friction decreased. Tightening the bearing was found to increase the friction considerably.

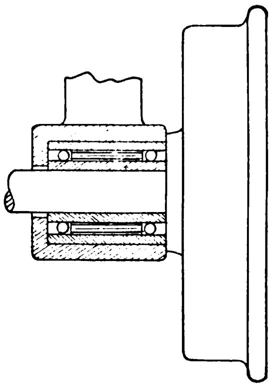


FIG. 236

Tests of the friction of steel rollers 1, 2, 3, and 4 in. in diameter are reported in the Trans. Am. Soc. C. E., August, 1894. The rollers were tested between plates $1\frac{1}{2}$ in. thick and 5 in. wide, arranged as shown in Fig. 237. Tests were made with the plates and rollers of cast iron, wrought iron, and steel. The friction P' for unit load P was found to be $\frac{.0063}{\sqrt{r}}$ for cast-iron rollers and plates, $\frac{.0120}{\sqrt{r}}$ for wrought iron, and $\frac{.0073}{\sqrt{r}}$ for steel, where r represents the radius of the roller in inches. The rollers were turned and the plates planed, but neither was polished.

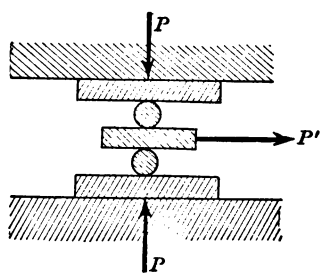


FIG. 237

163. Ball Bearings. — For high speeds and light or moderate loads the friction is much reduced by the use of hardened steel balls instead of the steel rollers. These bearings are now used on all classes of machinery, giving a much greater efficiency except for heavy loads. The principal objection to the ball bearing seems to be due to the fact that there is

so little area of contact between the balls and bearing plates. This gives rise to very high stresses over these areas, and consequently a considerable deformation of the balls. When the ball has been changed from its spherical form, it is no longer free to roll, and the friction increases rapidly. Some authorities consider a load of from 50 to 150 lb. sufficient for balls varying in size from $\frac{1}{4}$ to $\frac{1}{2}$ inch in diameter. Figure 238 illustrates a type of bearing used for shafts, and Fig. 239 a type used for thrust blocks.

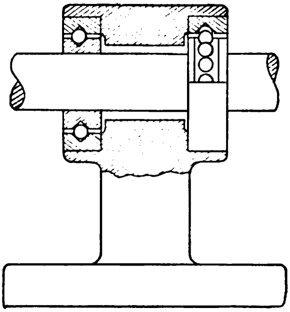


FIG. 238

The conclusions reached by Goodman from a series of tests on bicycle bearings (Proc. Inst. C. E., Vol. 89) are as follows:

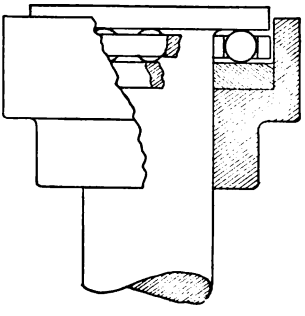


FIG. 239

(1) *The coefficient of friction of ball bearings is constant for varying loads, hence the frictional resistance varies directly as the load.*

(2) *The friction is unaffected by a change of temperature.*

The bearings were oiled before starting the tests. The coefficient of friction for ball bearings was found to be rather higher than for plain bearings with bath lubrication, but lower than for ordinary lubrication. Ball bearings will also run easily with a less supply of oil. The following table gives the results of

tests of ball bearings. The bearings were oiled before starting, and the tests were run at a temperature of 68° F.

Load on Bearing in Lb.	19 Revolutions per Min.		157 Revolutions per Min.		350 Revolutions per Min.	
	Coeff. friction	Friction, lb.	Coeff. friction	Friction, lb.	Coeff. friction	Friction, lb.
10	.0060	.06	.0105	.10	.0105	.10
20	.0045	.09	.0067	.13	.0120	.24
30	.0050	.15	.0050	.15	.0110	.33
40	.0052	.21	.0052	.21	.0097	.39
50	.0054	.27	.0054	.27	.0090	.45
60	.0050	.30	.0055	.33	.0075	.45
70	.0049	.34	.0054	.38	.0068	.47
80	.0048	.38	.0062	.49	.0060	.48
90	.0050	.45	.0068	.61	.0060	.54
100	.0058	.58	.0069	.69	.0057	.57
110	.0054	.59	.0065	.71	.0060	.66
120	.0055	.66	.0075	.90	.0057	.68
130	.0058	.75	.0078	1.01	.0062	.81
140	.0056	.78	.0077	1.08	.0060	.84
150	.0060	.90	.0083	1.24	.0062	.93
160	.0075	1.20	.0081	1.29	.0058	.93
170	.0079	1.34	.0078	1.33	.0055	.93
180	.0079	1.42	.0078	1.40	.0053	.95
190	.0087	1.65	.0076	1.44	.0054	1.03
200	.0090	1.80	.0081	1.62	.0060	1.20

Another series of tests, run with a constant load on the bearing of 200 lb. and a temperature of 86° F., shows the variation of the coefficient of friction with the speed. It is seen that as the speed *increased* the coefficient and the friction *decreased*. The preceding table, however, shows, *for loads below* 175 *lb., an increase in the coefficient with increase in speed.* In particular, this table shows that for

loads below 80 lb. the coefficient increased with increase of speed; for loads between 90 and 175 lb. it increased when the speed was 150 r.p.m. and decreased when it was 350 r.p.m. Beyond 175 lb. the coefficient increased.

Revolutions per Minute	Coefficient Friction	Friction Pounds
15	.00735	1.47
93	.00465	.93
175	.00375	.75
204	.00345	.69
280	.00300	.60

It seems from the data given that the first conclusion of Goodman's should be changed to read: the *coefficient of friction of ball bearings is constant for varying loads, up to a certain limit, beyond which it increases with increase of load.* This limit is about 150 lb. in the tests reported.

Tests on ball bearings designed for machinery subjected to heavy pressures have been made in Germany. (See *Zeitschrift des Vereins deutsche Ingenieure*, 1901, p. 73.) It was found that at speeds varying from 65 to 780 revolutions per minute, where the bearing was under pressures varying from 2200 lb. to 6600 lb., the coefficient of friction varied little and averaged .0015.

Tests of ball bearings made by Stribeck and reported by Hess (Trans. Am. Soc. M. E., Vol. 28, 1907) give rise to the following conclusions: (*a*) the load that may be put upon a bearing is given by the formula

$$P=\frac{cd^2n}{11.02},$$

ment of the cord on the small drum has traveled from rest at A to A' through 90°. Neglect the friction at B.

182. The Compound Pendulum.—A body rotating under the action of gravity about a horizontal axis is called a *compound pendulum*.

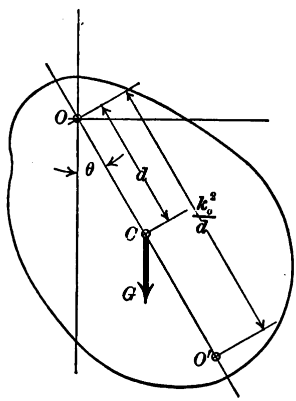


FIG. 261

Let Fig. 261 represent a compound pendulum rotating about an axis through O perpendicular to the plane of the paper. Let the distance from the axis of rotation to the center of gravity of the body be d.

Applying D'Alembert's principle, we have

$$-Gd\sin\theta = \alpha I_0,$$

or, since

$$I_0 = \frac{G}{g}k_0^2,$$

$$\alpha = -\frac{gd}{k_0^2}\sin\theta.$$

It was shown in Art. 120 that the tangential acceleration of a simple pendulum of length l is

$$a_t = -g\sin\theta,$$

and hence its angular acceleration is $\frac{a_t}{l}$, or

$$\alpha = -\frac{g}{l}\sin\theta.$$

Hence the angular motion of a compound pendulum is exactly the same as that of a simple pendulum whose

length, l, is such that

$$\frac{d}{k_0^2}=\frac{1}{l},$$

or

$$l=\frac{k_0^2}{d}.$$

This value, $\frac{k_0^2}{d}$, is called the length of the equivalent simple pendulum.

The time of a small vibration of a compound pendulum is therefore

$$t=\pi\sqrt{\frac{k_0^2}{gd}}.$$

The point, O, of the pendulum, about which rotation takes place, is called the *center of suspension.*

The point O' on OC such that $OO'=\frac{k_0^2}{d}$ is called the *center of oscillation.* It is that point at which the whole mass might be concentrated without changing the time of vibration, the center of suspension remaining the same.

Problem 420. A board 4 ft. by 1 ft. by 1 in. vibrates about an axis perpendicular to the 4 ft. by 1 ft. face through a point of the board 18 in. from the center. Find the time of vibration for small oscillations, and the length of the equivalent simple pendulum.

Problem 421. Show that the time of vibration is the same (for equal angles of vibration) for all parallel axes of suspension that are at equal distances from the center of gravity of the body.

Problem 422. A cast-iron sphere whose radius is 6 in. vibrates as a pendulum about a tangent line as an axis. Find the period of vibration and the length of a simple pendulum having the same period. Locate the center of oscillation.

183. Centers of Oscillation and Suspension Interchangeable. — If $\bar{k}$ is the radius of gyration of the body about the axis through the center of gravity parallel to the axis of suspension, then

$$k_0^2 = \bar{k}^2 + d^2.$$

Therefore
$$l = \frac{\bar{k}^2 + d^2}{d},$$

or
$$d(l - d) = \bar{k}^2.$$

In this formula d and $l - d$ enter in exactly the same way. It follows therefore that if O' were taken as a center of suspension, O would be the center of oscillation. That is, *in a compound pendulum the centers of oscillation and suspension are interchangeable.* This is easily verified experimentally.

Problem 423. In a plane through the center of gravity, C, of a body circles of radii r and $\frac{\bar{k}^2}{r}$ are drawn with centers at C, where $\bar{k}$ is the radius of gyration about a gravity axis perpendicular to the given plane. Show that the time of vibration is the same for all axes of suspension perpendicular to the given plane and passing through a point on the circumference of either circle.

184. Experimental Determination of Moment of Inertia. — The computation of the moment of inertia of many bodies is a difficult matter. It is often convenient, therefore, to use an experimental method in dealing with such bodies. The compound pendulum furnishes a means whereby such determinations may be made. From Art. 182, we find that the time of vibration of a compound pendulum is

$$t = \pi \sqrt{\frac{k_0^2}{gd}}.$$

This may be written

$$k_0^2 = \frac{t^2}{\pi^2} gd.$$

Multiplying both sides by M, the mass of the body, we have

$$I_0 = M\frac{t^2}{\pi^2} gd = G\frac{t^2}{\pi^2} d.$$

It thus appears that if d, the distance from O to the center of gravity, is known (the center of gravity may be located by balancing over a knife edge) and also the weight G, and the body be allowed to swing as a pendulum about O as an axis, t may be determined, giving I_0.

If I_g be desired, it may be determined from the formula (see Art. 64),

$$I_g = I_0 - Md^2.$$

Problem 424. The connecting rod of a high-speed engine tapers regularly from the cross-head end to the crank-pin end. Its length is 10 ft., its cross section at the large end 5.59″ × 12.58″ and at the cross-head end 5.59″ × 8.39″. Neglecting the holes at the ends, the center of gravity is 64 in. from the cross-head end. The rod is made of steel and vibrates as a pendulum about the cross-head end in 1.3 sec. Compute its moment of inertia about the gravity axis.

The student should take such a connecting rod as the one in the preceding problem, or other body, and by swinging it as a pendulum find its period of vibration. Compute the moment of inertia about the axis of suspension and about the gravity axis.

185. Determination of *g*. — From the preceding article we see that

$$g = \frac{k_0^2\pi^2}{dt^2} = \frac{I_0\pi^2}{Mt^2d}.$$

This relation enables us to determine g, as soon as we know I_0, M, and d, by determining the time of vibration about the point O. It is evident that $\frac{I_0\pi^2}{Md}$ is a constant for the body, when the axis is through O, and that when once determined accurately, the pendulum might be used to determine g for any locality.

This constant, $\frac{I_0\pi^2}{Md}$, is known as the *pendulum constant.*

Problem 425. A round rod of steel 6 ft. long is made to swing as a pendulum about an axis tangent to one end and perpendicular to its length. The rod is 1 in. in diameter. Determine the pendulum constant.

Problem 426. The center of gravity of a connecting rod 5 ft. long is 3 ft. from the cross-head end. The rod is vibrated as a pendulum about the cross-head end. It is found that 50 vibrations are made in a minute. Find the radius of gyration with respect to the cross-head end.

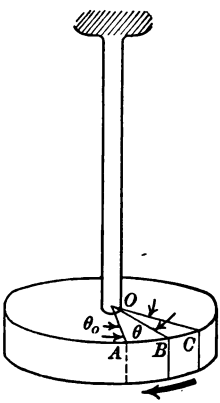


FIG. 262

186. The Torsion Balance. — A torsion balance consists of a body suspended by a slender rod or wire attached rigidly to the body and at the point of support, the center of gravity of the body lying in the line of the wire (Fig. 262).

Let OA be the neutral position of a line in the surface of the disk used as a torsion pendulum (Fig. 262).

Let the disk be turned through an angle θ_0 and released. The wire is

then twisted and exerts a twisting moment on the body, tending to restore it to the neutral position. Experiment shows that this twisting moment exerted by the wire is proportional to the angle of twist. Hence, since

$$\Sigma \text{ mom} = I\alpha,$$

we have for the motion of the disk, starting from the position where $\theta = \theta_0$,

$$-C\theta = I\omega \frac{d\omega}{d\theta}, \tag{1}$$

the minus sign being used since $\omega \dfrac{d\omega}{d\theta}$ is the acceleration counter-clockwise and $C\theta$ represents the clockwise twisting moment.

$$\therefore I\omega d\omega = -C\theta d\theta.$$

Integrating,

$$I\frac{\omega^2}{2} = -C\frac{\theta^2}{2} + C_1.$$

When $\quad t = 0,\ \theta = \theta_0,\ \omega = 0. \quad \therefore C_1 = C\dfrac{\theta_0^2}{2}.$

$$\therefore I\omega^2 = C(\theta_0^2 - \theta^2). \tag{2}$$

This equation shows that the angular velocity of the body is the same for negative values of θ as for the numerically equal positive values. The motion is therefore periodic, the body vibrating through equal angles on both sides of the neutral position.

Considering the motion from $\theta = \theta_0$ to $\theta = 0$, equation (2) may be written

$$\omega = \frac{d\theta}{dt} = -\sqrt{\frac{C}{I}}\sqrt{\theta_0^2 - \theta^2}.$$

$$\therefore \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}} = -\sqrt{\frac{C}{I}} \cdot dt.$$

Integrating,

$$\sin^{-1}\frac{\theta}{\theta_0} = -\sqrt{\frac{C}{I}} \cdot t + C_2.$$

When $t = 0$, $\theta = \theta_0$. $\therefore C_2 = \sin^{-1} 1$.

Choosing $\frac{\pi}{2}$ as the value of $\sin^{-1} 1$, we have

$$\sin^{-1}\frac{\theta}{\theta_0} = \frac{\pi}{2} - \sqrt{\frac{C}{I}} \cdot t.$$

This equation shows that as t increases $\sin^{-1}\frac{\theta}{\theta_0}$ must decrease. Hence, since we chose $\sin^{-1}\frac{\theta}{\theta_0} = \frac{\pi}{2}$ when $\theta = \theta_0$, the angle $\frac{\theta}{\theta_0}$ must decrease from $\frac{\pi}{2}$ and will therefore reach zero when $\theta = 0$.

Hence $$t = \frac{\pi}{2}\sqrt{\frac{I}{C}} \qquad \text{when } \theta = 0.$$

Therefore the time for the body to make a complete swing, one way, is

$$t = \pi\sqrt{\frac{I}{C}}.$$

If m_1 is the twisting moment exerted by the wire when twisted through an angle θ_1, $m_1 = C\theta_1$, and the expression for the period becomes

$$t = \pi\sqrt{\frac{I\theta_1}{m_1}}.$$

The time of vibration is independent of the initial angular displacement.

187. Determination of the Moment of Inertia of a Body by Means of the Torsion Balance. — The moment of inertia of a body may be found by suspending it by a wire and observing the time of vibration when used as a torsion balance. The constant $C = \frac{m_1}{\theta_1}$ is a constant of the wire or rod and depends upon the material and diameter. Knowing this constant, it would only be necessary to determine the period of vibration in order to find I.

For practical purposes, however, it is desirable to eliminate from consideration the value $\frac{m_1}{\theta_1}$. For this purpose suppose the disk provided with a suspended platform rigidly attached as shown in cross section in Fig. 263. Let t be its time of vibration and I its moment of inertia about the axis of suspension. Now place on the disk two equal cylinders H in such a way that their center of gravity is the axis of suspension. Let t_1 be the period of vibration of the cylinders and support and I_1 their moment of inertia.

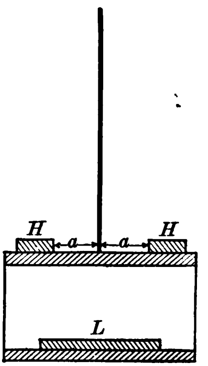


FIG. 263

Then $\frac{t^2}{t_1^2} = \frac{I}{I_1}$. The moment of inertia of the two cylinders with respect to the axis of rotation is known; call it I_2. Then

$$I_1 = I + I_2,$$

so that

$$I = I_2 \frac{t^2}{t_1^2 - t^2}.$$

This gives the moment of inertia of the torsion balance, which, of course, is a constant.

The moment of inertia of any body L may now be determined by placing the body on the suspended platform with its center of gravity in the axis of rotation and noting the time of vibration. Calling the time of vibration of the body L and the balance t_3 and their moment of inertia I_3, we have

$$\frac{t^2}{t_3^2} = \frac{I}{I_3}.$$

Let the moment of inertia of L itself be I_4, so that

$$I_3 = I + I_4.$$

Then
$$I_4 = I\frac{t_3^2 - t^2}{t^2}.$$

This method may be used in finding the moment of inertia of non-homogeneous bodies, provided the center of gravity be placed in the axis of rotation.

Problem 427. The moment of inertia of a torsion balance is 6300, where the units of mass, space, and time are respectively the gram, centimeter, and second, and its time of vibration 20 sec. The body L consists of a homogeneous cast-iron disk 3 in. in diameter and 1 in. thick. Find the time of vibration of the balance when L is in place. Compute the moment of inertia of the disk.

Problem 428. The same balance as that used in the preceding problem is loaded with a body L, and the time of vibration is found to be 30 sec. Determine the moment of inertia of L.

188. Rotating Body under the Action of no Forces. — If a body is rotating about a fixed axis, the axis of z, and is acted upon by no forces, the equations of Art. 180 become

$$0 = -\omega^2\bar{x}M - \alpha\bar{y}M, \tag{1}$$
$$0 = -\omega^2\bar{y}M + \alpha\bar{x}M, \tag{2}$$
$$0 = 0, \tag{3}$$
$$0 = \omega^2\int yzdM - \alpha\int xzdM, \tag{4}$$
$$0 = -\omega^2\int xzd\,M - \alpha\int yzdM, \tag{5}$$
$$0 = \alpha I_z. \tag{6}$$

From these equations it follows that

$$\alpha = 0, \quad \int xzdM = 0, \quad \int yzdM = 0, \quad \bar{x} = 0, \quad \bar{y} = 0.$$

Hence, *if a body is rotating about a fixed axis and no forces are acting on the body,*

(1) *the angular velocity is constant,*

(2) *the axis of rotation passes through the center of gravity,*

(3) *if the axis of rotation is chosen as the axis of z, then*

$$\int xzdM = 0, \text{ and } \int yzdM = 0.$$

(When $\int xzdM = 0$ and $\int yzdM = 0$, the z-axis is a *principal axis* of the body. It can be shown that there are three lines at right angles to each other through any point of the body which are principal axes, and that the ellipsoid of inertia of the body for that point has its axes lying on these three lines. Compare Art. 77.)

189. Rotation of a Body about a Fixed Axis with Constant Angular Velocity. — If the angular velocity is constant, $\alpha = 0$, and the equations of Art. 180 take simpler forms.

In addition, at a given instant, let the x-axis be chosen through the center of gravity of the body. Then $\bar{y} = 0$, and equations (1) to (6) of Art. 180 become

$$\Sigma X_i = -\omega^2 \bar{x} M, \tag{1}$$

$$\Sigma Y_i = 0, \tag{2}$$

$$\Sigma Z_i = 0, \tag{3}$$

$$\Sigma \,\text{mom}_{ix} = \omega^2 \int yz dM, \tag{4}$$

$$\Sigma \,\text{mom}_{iy} = -\omega^2 \int xz dM, \tag{5}$$

$$\Sigma \,\text{mom}_{iz} = 0. \tag{6}$$

The first equation shows that unless $\bar{x}$ is zero there is a resultant of all the x-components of the impressed forces and that this resultant is

$$R_x = -\omega^2 \bar{x} M.$$

Equations (2) and (3) show that the y-components of the impressed forces either have a resultant zero or else form a couple, and the same of the z-components.

A particular case is that where

$$\int xz dM = 0, \text{ and } \int yz dM = 0,$$

as, for example, where the xy-plane is a plane of symmetry. Suppose in this case the only impressed forces are the weights and the reactions of the supports, as indicated in Fig. 264.

Equations (1) to (6) then become

$$P_x + P'_x = -\omega^2 \bar{x} M, \tag{1'}$$

$$P_y + P'_y = 0, \tag{2'}$$

$$P_z - W = 0, \tag{3'}$$

$$bP_y - aP'_y = 0, \tag{4'}$$

$$aP'_x + \bar{x}W - bP_x = 0, \tag{5'}$$

$$0 = 0. \tag{6'}$$

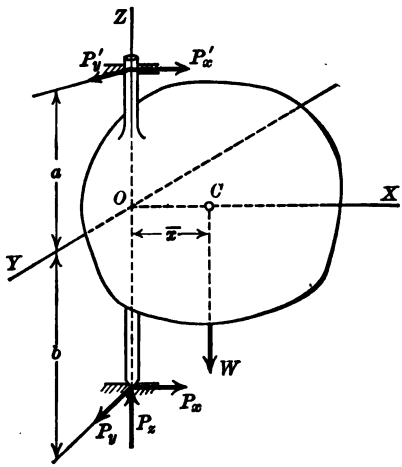


FIG. 264

From (2′) and (4′), $P_y = 0,\ P'_y = 0.$

From (3′), $P_z = W.$

From (1′) and (5′), $P_x = \dfrac{\bar{x}}{a+b}W - \dfrac{a}{a+b}\bar{x}\omega^2 M,$

$$P'_x = -\frac{\bar{x}}{a+b}W - \frac{b}{a+b}\bar{x}\omega^2 M.$$

If the weight W were counterbalanced by an equal upward force acting through C, the only forces acting on the axis would be

$$P_x = -\frac{a}{a+b}\bar{x}\omega^2 M, \quad P'_x = -\frac{b}{a+b}\bar{x}\omega^2 M.$$

The resultant of these forces is a single force,

$$R_x = -\bar{x}\omega^2 M,$$

and if its distance above the xy-plane is z',

$$\begin{aligned} z'R_x &= aP'_x - bP_x \\ &= -\frac{ab}{a+b}\bar{x}\omega^2 M + \frac{ab}{a+b}\bar{x}\omega^2 M \\ &= 0. \end{aligned}$$

Therefore, when the xy-plane is a plane of symmetry, or when $\int xzdM = 0$ and $\int yzdM = 0$, the effect of the rotation of the body about the axis of z is to cause an outward pull on the axis of rotation in a line through the center of gravity perpendicular to the axis of rotation and of the same value, $M\bar{x}\omega^2$, that would be caused by concentrating the whole mass at the center of gravity.

Problem 429. A wheel of weight 100 lb. is mounted on an axle and is off center $\frac{1}{2}$ inch, the plane of the wheel being perpendicular to the axis. Find the force tending to bend the shaft when the wheel is making 200 r. p m.

Problem 430. A thin rod, 2 ft. long, weighing 5 lb. is attached to an axle at an angle of 60°. Find the outward force in magnitude and position in the plane of the rod and axle due to the rotation when making 150 r. p. m. If the rod is joined to the axle 2 ft. and 1 ft. from the supports (Fig. 265), find the reaction at the supports due to the rotation.

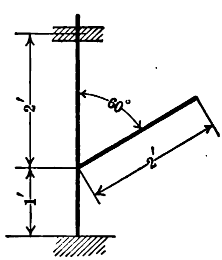


FIG. 265

Problem 431. A steel disk 3 ft. in diameter and 1 in. thick is not perpendicular to the axis of rotation, but is out of true by $\frac{1}{100}$ of its radius. Find the twisting couple introduced tending to make the shaft wobble.

Problem 432. A grindstone weighing 200 lb., of radius 2 ft., is making 100 r. p. m. Find the force with which one half of the stone pulls on the other half.

Problem 433. Neglecting the weight and the tension in the spokes of a rotating fly-wheel, prove that the tension in the rim is

$$P = \frac{\omega^2 r^2 \gamma F}{g},$$

where ω = angular velocity, γ = the heaviness of the material, F = the area of the cross section of the rim, and r = mean radius of the rim (Fig. 266).

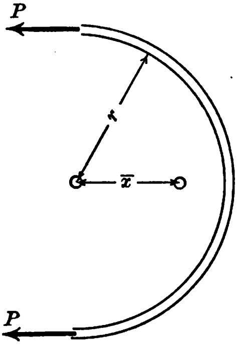


FIG. 266

Problem 434. In the preceding problem, suppose r = 6 ft., F = 10 in. by 4 in., and the wheel is made of cast iron. If the tensile strength of the material is 25,000 lb. per square inch, what speed would be attained before the wheel bursts?

Problem 435. Show that if P is the pressure of the side rod of a locomotive on the crank pin, r the radius of the crank-pin circle, r' the radius of the driver, and v the velocity of the train,

$$2P - G = \frac{Grv^2}{gr'^2}$$

when the side rod is in the lowest position (Fig. 267).

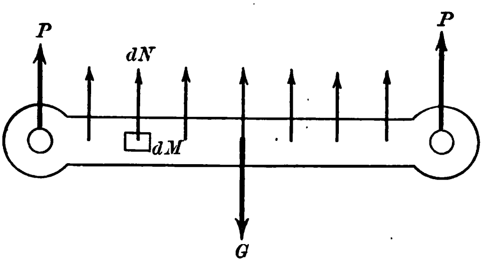


FIG. 267

Problem 436. Suppose the velocity of a locomotive to be 90 mi. per hour, the radius of the crank-pin circle 20 in., the radius of the drive wheel 40 in., and the weight of the side rod 400 lb. Find the pressure on the crank pins due to the rotation alone.

190. Rotation of a Locomotive Drive Wheel. — The drive wheel of a locomotive (Fig. 268) may be considered for the present as rotating about a fixed axis. We shall consider the effect of the weight of the counterbalance on the tire due to rotation only, on the assumption that the tire carries all the weight of the counterbalance.

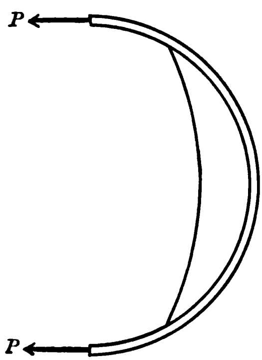


FIG. 268

NOTE. It is to be understood that the wheel center carries part of the weight of the counterbalance, but a complete solution of the problem of the drive wheel is beyond the scope of this book. The above assumption is therefore made.

Let M be the mass of $\frac{1}{2}$ of tire and $\bar{\rho}$ the distance of its center of gravity from the center of wheel. Let M_1 be the mass of the counterbalance, and $\bar{\rho}_1$ the distance of its center of gravity from the center of wheel.

Then $$2P = \omega^2 (M\bar{\rho} + M_1\bar{\rho}_1).$$

In particular, suppose the diameter of the tread of the tire to be 80 in., distance of the center of gravity of $\frac{1}{2}$ of tire from center 27 in., and mass of $\frac{1}{2}$ of tire 21. The mass of the counterbalance is 20, and the distance of its center of gravity from the center of the wheel 29 in. Substituting these values, we get

$$2P = \omega^2 \left[21\left(\tfrac{27}{12}\right) + 20\left(\tfrac{29}{12}\right)\right] = 95.6\,\omega^2.$$

If now we know the speed of rotation of the wheel so that ω is known, we may determine P. Let us take ω corresponding to a speed of train of 60 mi. per hour.

This gives $\omega = 26.4$ radians per second and

$$P = 33{,}300 \text{ lb.}$$

Problem 437. If the area of the cross section of the tire is 20 sq. in., find the stress on the metal due to rotation about the axis under the above assumption.

If the allowable stress on the metal is 20,000 lb. per square inch, find the speed of train to cause this stress.

191. Standing and Running Balance of a Shaft. — Let weights, W_1, W_2, W_3, W_4, etc., be attached to a shaft, the distances of their centers from the center of the shaft being r_1, r_2, r_3, r_4, etc. (Fig. 269), it being assumed that

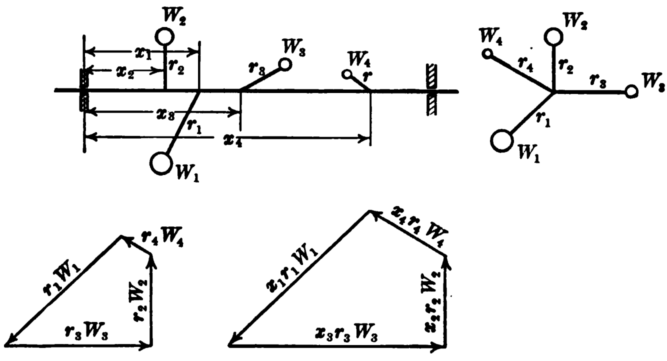


FIG. 269

each weight is symmetric with respect to a plane through its center and perpendicular to the axis of the shaft, so that the pull of the weight on the shaft due to rotation is through the center of gravity of the weight (Art. 189).

If the shaft is balanced when at rest in any horizontal position, the sum of the moments of the weights about the axis must be zero for any position. The moment of

W_1 is W_1 times the horizontal component of r_1, or it is the horizontal component of $W_1 r_1$ laid off along r_1. The weights will then be in standing balance when, and only when, the sum of the horizontal components of the vectors $r_1 W_1$, $r_2 W_2$, $r_3 W_3$, etc., is zero for any horizontal position of the shaft. Hence the vector polygon

$$r_1 W_1 + r_2 W_2 + r_3 W_3 + \cdots$$

must close.

For running balance the reactions at each support, due to the centrifugal pull of the weights, must annul or the shaft will tend to wobble. These forces due to rotation are along the radii (Art. 189), and if x_1, x_2, x_3, $\cdots$ are the distances along the shaft of the weights from one bearing, the reactions at the other bearing are $\frac{x_1}{l} r_1 \omega^2 M_1$, $\frac{x_2}{l} r_2 \omega^2 M_2$, $\cdots$, where l is the distance between the supports, and M_1, M_2, M_3, etc., are the masses of the weights.

These reactions are therefore proportional to $x_1 r_1 W_1$, $x_2 r_2 W_2$, etc., and are in the directions of r_1, r_2, etc., respectively. In order to balance, the polygon of vectors

$$x_1 r_1 W_1 + x_2 r_2 W_2 + \cdots$$

must therefore close.

In order that the reactions due to rotation balance at the other bearing, the vector polygon

$$(l - x_1) r_1 W_1 + (l - x_2) r_2 W_2 + \cdots$$

must close. But since the vectors may be taken in any order, this polygon may be plotted in the order

$$l(r_1 W_1 + r_2 W_2 + r_3 W_3 + \cdots) +$$
$$(x_1 r_1 W_1 + x_2 r_2 W_2 + x_3 r_3 W_3 + \cdots)$$

which clearly will close when the two polygons already considered close.

There will then be standing and running balance when two closed polygons can be formed of vectors $r_1 W_1$, $r_2 W_2$, $r_3 W_3$, etc., and $x_1 r_1 W_1$, $x_2 r_2 W_2$, $x_3 r_3 W_3$, etc., respectively, the vectors having the directions of the perpendiculars from the center line of the axle to the centers of gravity of the corresponding weights.

Problem 438. In Fig. 269 assume $W_1 = 6$, $W_2 = 5$, $W_3 = 4$; $r_1 = 4.3$, $r_2 = 3$, $r_3 = 6$, $r_4 = 2$; angle between r_1 and $r_3 = 135°$, angle between r_3 and $r_2 = 90°$; $x_1 = 1.5$, $x_2 = 1$. Find x_3 and the value and position of W_4 for running balance.

Problem 439. Show that two weights, given in position and magnitude in the same plane perpendicular to the axis of rotation, can be balanced for running by a third given weight in that plane. Show how to determine the position of this third weight, and show that the condition for standing balance is the same as for running balance in the case of weights all in the same plane perpendicular to the axis of rotation.

Problem 440. If two weights are attached to the axis in the same plane containing the axis, show how to find the position of a third given weight that will balance the two given weights for running and standing balance. If balanced for standing, will they be balanced for running?

Problem 441. Show that three weights cannot be balanced for running unless they either lie all in one plane perpendicular to the axis of rotation or else all in a plane containing the axis.

Problem 442. Show that four weights and the arms of three of them being given and two of the weights definitely located on the axis, the other weights may be placed so as to form standing and running balance, and show how the location of the two weights would be determined. Could this be done in more than one way?

192. Rotation of Flywheel of Steam Engine. – In Fig. 270 let a belt run horizontally over a flywheel, the tensions being P_1 and P_2 where $P_2 > P_1$. The effective steam pressure is P, N' the pressure of the guides on the cross-head. It is normal if friction is neglected.

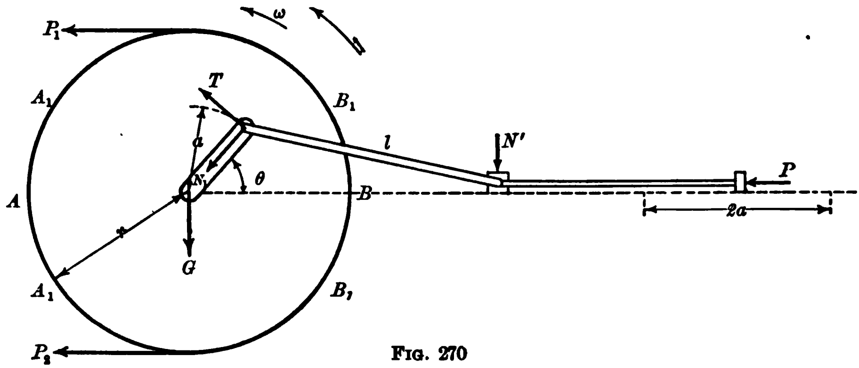


FIG. 270

The pressure on the crank pin is resolved into tangential and radial components, T and N_1.

From the relation,

$$\Sigma \text{ moms} = \alpha I,$$

we have for the motion of the wheel

$$Ta - (P_2 - P_1)r = \alpha I. \qquad (1)$$

It will be assumed that the resistance of the machinery, as shown by $P_2 - P_1$, is constant.

When $T = (P_2 - P_1)\frac{r}{a}$, $\alpha = 0$, and the angular velocity, ω, is either a maximum or a minimum.

When $\theta = 0$ (Fig. 270), $T = 0$, α is negative, and ω is

therefore decreasing. Hence ω has a minimum value in the first quadrant at B_1 where $T = (P_2 - P_1)\frac{r}{a}$.

From B_1 the angular velocity increases as long as T remains greater than $(P_2 - P_1)\frac{r}{a}$. In the second quadrant as T decreases toward zero at the dead point A, there is a point A_1 where $T = (P_2 - P_1)\frac{r}{a}$ at which therefore ω has a maximum value. In the same way the points A_1 and B_1 in the third and fourth quadrants correspond to minimum and maximum values of ω.

Equation (1) may be written

$$I\omega d\omega = Tad\theta - (P_2 - P_1)rd\theta. \qquad (1')$$

If s and s' are the arcs passed over in the crank-pin circle and the flywheel circle respectively, then

$$ad\theta = ds \text{ and } rd\theta = ds',$$

and equation (1′) becomes on integrating from an initial value ω_0 to any value ω,

$$\tfrac{1}{2} I(\omega^2 - \omega_0^2) = \int_{s_0}^{s} Tds - (P_2 - P_1)(s' - s_0').$$

Since the work done on one end of the connecting rod equals the work done by the other, neglecting its own change in kinetic energy,

$$\int_{s_0}^{s} Tds = \int_{x_0}^{x} Pdx,$$

and hence

$$\tfrac{1}{2} I(\omega^2 - \omega_0^2) = \int_{x_0}^{x} Pdx - (P_2 - P_1)(s' - s_0').$$

The approximate value of $\int_{x_0}^{x} P dx$ may be found by reading from the indicator card the values of P for successive values of x between the limits x and x_0.

A different treatment of the above equation may be obtained by considering that the pressure of steam in the cylinder is constant and equal to P' up to the point of cut-off and that beyond this point the pressure varies inversely as the volume. If we assume P constant and equal to P' to the cut-off, then the limits of integration will be regarded accordingly, and we may write

$$\tfrac{1}{2} I(\omega_1^2 - \omega_0^2) = P'(x_1 - x_0) - (P_2 - P_1)(s_1' - s_0'),$$

where x_1 is the value of x at the cut-off.

Beyond the cut-off P varies inversely as the volume of steam in the cylinder, or

$$P = \frac{x_1}{x} P'.$$

Then from the point of cut-off to any value of x,

$$\tfrac{1}{2} I(\omega^2 - \omega_1^2) = x_1 P' \int_{x_1}^{x} \frac{dx}{x} - (P_2 - P_1)(s' - s_1')$$

$$= x_1 P' \log_e\left(\frac{x}{x_1}\right) - (P_2 - P_1)(s' - s_1').$$

If P_1' is the mean effective pressure, we have, considering the work done on the flywheel for the motion from B to A, neglecting friction and the change in kinetic energy of the connecting rod,

$$\tfrac{1}{2} I(\omega_A^2 - \omega_B^2) = 2\,aP_1' - (P_2 - P_1)\pi r.$$

Problem 443. Suppose the mean effective steam pressure is 16,000 lb., the radius of the crank-pin circle 18 in., and the radius of the fly-

wheel 3 ft. If $P_2 - P_1 = 500$ lb., $I_s = 2000$, and $\omega_B = 2\pi$ radians per second, find ω_A.

Problem 444. The flywheel in the above problem has a velocity $\omega_A = 6\pi$ radians per second. What constant resistance $(P_2 - P_1)$ will change this to 2π radians per second in 100 revolutions?

Problem 445. Find the values of θ for which ω is maximum and minimum in the above problem.

Problem 446. In Problem 443 find the value of ω when the wheel has turned through 90° from B.

Problem 447. If the total pressure in the above problem is 20,000 lb. up to the cut-off at $\frac{1}{3}$ the stroke, find ω_A.

193. Rotation and Translation. — Let O be a point in a body having plane motion, *i.e.* translation and rotation in a plane, and let the angular velocity and angular acceleration of the body be ω and α respectively (Fig. 271).

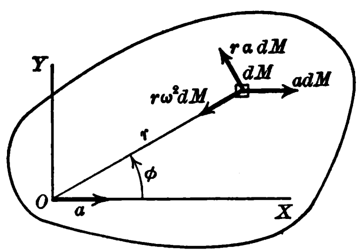


FIG. 271

At any instant choose axes with origin at O and the x-axis along the vector of a, the linear acceleration of the point O. Any element of the body, distant r from O, then has the acceleration a parallel to the x-axis, a tangential acceleration $r\alpha$, and a radial acceleration $r\omega^2$, and the corresponding effective forces acting on dM are adM, $r\alpha dM$, and $r\omega^2 dM$.

Applying D'Alembert's principle, taking the z-axis through O perpendicular to the plane of motion,

$$\Sigma X_i = \int (adM - r\omega^2 dM \cos\phi - r\alpha dM \sin\phi),$$

$$\Sigma Y_i = \int (r\alpha dM \cos\phi - r\omega^2 dM \sin\phi),$$

$$\Sigma \text{mom}_{is} = \int (r^2 \alpha dM - y a dM),$$

which reduce to

$$\Sigma X_i = Ma - \alpha \bar{y} M - \omega^2 \bar{x} M, \quad (1)$$

$$\Sigma Y_i = \alpha \bar{x} M - \omega^2 \bar{y} M, \quad (2)$$

$$\Sigma \text{mom}_{is} = \alpha I_s - a \bar{y} M. \quad (3)$$

194. The Connecting Rod. — The connecting rod of an engine has a circular motion at one end while the other end moves backward and forward in a straight line. We shall consider the motion relative to the engine and shall assume that the flywheel is of sufficient weight to give the crank a motion sensibly uniform. It will be convenient to regard the motion of the connecting rod as consisting of a rotation about the crosshead end while that end is moving in a straight line.

In Fig. 272 let A be the crosshead and O the center of the crank-pin circle. Let l be the length of the connecting rod, and r the radius of the crank-pin circle. If we neglect friction, the only forces acting on the connecting

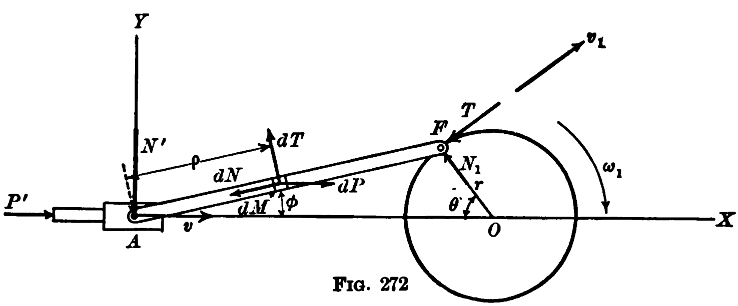


FIG. 272

rod at A are N', the pressure of the guides, and P', the pressure exerted by the piston rod. The force exerted on the connecting rod by the crank pin has been resolved into its normal and tangential components N_1 and T, respectively. Suppose ω_1 to be the constant angular velocity of the crank, and suppose the angular velocity of the rod to be represented by ω and the angular acceleration by α. Let a be the linear acceleration of the crosshead.

The equations of the preceding article applied to the connecting rod become

$$P' - T\sin\theta - N_1\cos\theta = Ma - \omega^2\bar{x}M - \alpha\bar{y}M, \quad (1)$$

$$-N' - G + N_1\sin\theta - T\cos\theta = -\omega^2\bar{y}M + \alpha\bar{x}M, \quad (2)$$

$$N_1 l\sin(\theta+\phi) - Tl\cos(\theta+\phi) - \tfrac{1}{2}Gl\cos\phi = \alpha I_z - a\bar{y}M. \quad (3)$$

Before these equations can be solved for the reactions, the values of ω, α, and a must be known. These values can be expressed in terms of ω_1.

From the figure,

$$r\sin\theta = l\sin\phi.$$

Differentiating,

$$r\cos\theta\frac{d\theta}{dt} = l\cos\phi\frac{d\phi}{dt}.$$

But

$$\frac{d\theta}{dt} = \omega_1, \text{ and } \frac{d\phi}{dt} = \omega.$$

$$\therefore r\omega_1\cos\theta = l\omega\cos\phi,$$

or

$$\omega = \frac{r\omega_1\cos\theta}{\sqrt{l^2 - r^2\sin^2\theta}}. \quad (4)$$

Differentiating again, we obtain

$$\alpha = \frac{d\omega}{dt} = -\frac{(l^2 - r^2)r\omega_1^2\sin\theta}{(l^2 - r^2\sin^2\theta)^{\frac{3}{2}}}. \quad (5)$$

To obtain a notice that any two points of the rod have velocities whose components along the rod are equal. Hence if v_1 is the velocity of the crank pin and v is the velocity of the crosshead,

$$v \cos \phi = v_1 \sin (\theta + \phi). \qquad \text{(Fig. 272)}$$

$$\therefore \; v = v_1 (\sin \theta + \cos \theta \tan \phi).$$

$$a = \frac{dv}{dt} = v_1 \left(\cos \theta \frac{d\theta}{dt} - \sin \theta \tan \phi \frac{d\theta}{dt} + \cos \theta \sec^2 \phi \frac{d\phi}{dt} \right)$$

$$= v_1 \left[\frac{\cos (\theta + \phi)}{\cos \phi} \omega_1 + \frac{\cos \theta}{\cos^2 \phi} \omega \right],$$

or

$$a = \frac{r\omega_1}{\cos^2 \phi} [\omega_1 \cos (\theta + \phi) \cos \phi + \omega \cos \theta]. \tag{6}$$

The value of ω from equation (4) may be substituted in equation (6) and thus the value of a determined.

Problem 448. The connecting rod given in Problem 424, Art. 184, is in use on an engine whose crank has a constant angular velocity of 26 radians per second. The length of the crank is 2 ft., the effective steam pressure on the piston is 16,000 lb. Use the values of M, I, $\bar{x}$, and $\bar{y}$ from Problem 424. Find the values of N', N_1, and T when $\theta = 30°$.

Problem 449. Show that ω has its greatest numerical value when $\theta = 0$ and $\theta = \pi$, and its least numerical value when $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. What are these greatest and least values?

Problem 450. Find what values of θ will make a a maximum or minimum. Locate the crosshead for these values.

Problem 451. Find values for T, N', N_1, and the resultant pressure on the crank pin when $\theta = \pi$ and when $\theta = 0$. Use the above data.

Problem 452. Assume a force of friction F acting on the crosshead, such that $F = .06\, N'$. In the above case when $\theta = 30°$, what is the value of F, N', N_1, and T?

Problem 453. Suppose the steam pressure zero, find T, N', N_1, and the resultant crank-pin pressure, if ω_1 is the same.

195. Angular Momentum or Moment of Momentum. — If the velocity of a particle is resolved into two components, one of which is parallel to a given line and the other in a plane perpendicular to the line, the product of the latter component, the perpendicular distance of this component from the given line, and the mass of the particle is called the *moment of momentum* or *angular momentum* of the particle with respect to the given line. Thus, if v_1 is the component of the particle in the plane perpendicular to the given line, d the distance of this component from the given line, and m the mass of the particle, then its

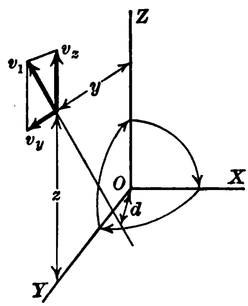


FIG. 273

$$\textit{angular momentum} = mv_1d. \qquad \text{(Fig. 273)}$$

If the given line be taken as the axis of x, the component v_1 may be resolved into components v_y and v_z parallel to the y- and z-axes, respectively. Then, since the moment of any vector is equal to the sum of the moments of its components, we have

angular momentum of the particle with respect to x-axis

$$= (yv_z - zv_y)m.$$

If dm is the element of mass of any body, then the angular momentum of the body about any line is defined to be the sum of the angular momenta of its particles about that line. Thus, representing the angular momenta of a body

about the x-, y-, and z-axes by h_x, h_y, h_z, respectively, we have

$$h_x = \int (yv_z - zv_y)dm, \quad h_y = \int (zv_x - xv_z)dm,$$
$$h_z = \int (xv_y - yv_x)dm.$$

196. Torque and Angular Momentum. — Let dm be the mass of an element of a body, and a_x, a_y, a_z, respectively, the components of its acceleration parallel to the axes.

Using D'Alembert's principle, we may equate the sum of the moments of the effective forces to the sum of the moments of the impressed forces.

The effective forces acting on dm are $a_x dm$, $a_y dm$, $a_z dm$, respectively parallel to the x-, y-, and z-axes. Hence calling the torques due to the impressed forces about the axes respectively T_x, T_y, T_z, we have

$$T_x = \int (ya_z - za_y)dm,$$
$$T_y = \int (za_x - xa_z)dm,$$
$$T_z = \int (xa_y - ya_x)dm.$$

Now $$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt},$$
$$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad a_z = \frac{dv_z}{dt}.$$

Also $$\frac{d}{dt}(yv_z - zv_y) = v_y v_z + ya_z - v_z v_y - za_y = ya_z - za_y.$$
$$\therefore \; T_x = \int \frac{d}{dt}(yv_z - zv_y)dm.$$

Since the derivative of a sum of terms is the sum of the derivatives of the separate terms, and the integral is the

limit of a summation, we may write this latter equation

$$T_x = \frac{d}{dt}\left[\int (yv_z - zv_y)dm\right],$$

or
$$T_x = \frac{dh_x}{dt}.$$

Likewise
$$T_y = \frac{dh_y}{dt},$$

and
$$T_z = \frac{dh_z}{dt}.$$

That is: *The moment of impressed forces about any axis is equal to the rate of change of the angular momentum about that axis.*

A corollary of this theorem is that when the moment of impressed forces acting on a body about an axis is constantly equal to zero, the angular momentum about that axis is constant.

197. Moment of Momentum of a Body with One Point Fixed in Terms of Angular Velocity. — If a body in motion has one point fixed, its motion at any instant is one of instantaneous rotation about an axis passing through the fixed point. Take the fixed point, O, of the body as origin of coördinates. The angular velocity about the instantaneous axis can be resolved into component angular velocities, ω_x, ω_y, ω_z, about the axes (Art. 132). The velocity v_x or $\frac{dx}{dt}$ is then due to rotation about OY and OZ. Figure 274 shows the component parallel to OX of the velocity due to rotation about OZ to be

$$-r_1\omega_z \cos\beta \text{ or } -y\omega_z.$$

In the same way the x-component of the velocity due to rotation about OY is $z\omega_y$.

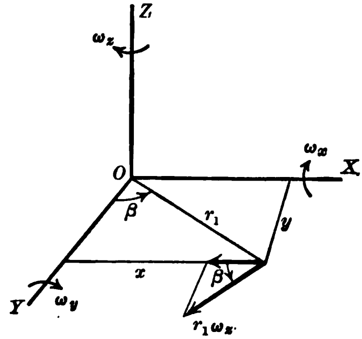


FIG. 274

$$\therefore \frac{dx}{dt} = z\omega_y - y\omega_z.$$

Similarly $$\frac{dy}{dt} = x\omega_z - z\omega_x,$$

and $$\frac{dz}{dt} = y\omega_x - x\omega_y.$$

Substituting these values in the expressions for h_x, we obtain

$$h_x = \int [y(y\omega_x - x\omega_y) - z(x\omega_z - z\omega_x)]dm$$

$$= \int (y^2 + z^2)\omega_x dm - \int xy\omega_y dm - \int xz\omega_z dm,$$

or $$h_x = I_x\omega_x - \omega_y \int xydm - \omega_z \int xzdm.$$

Similarly, $$h_y = I_y\omega_y - \omega_z \int yzdm - \omega_x \int yxdm,$$

and $$h_z = I_z\omega_z - \omega_x \int zxdm - \omega_y \int zydm.$$

The only body whose motion we shall study in this book is a uniform body in the form of a solid of revolution. If one of the axes be taken as the axis of the body, then all of the products of inertia vanish and the expressions for the angular momenta become

$$h_x = I_x\omega_x, \quad h_y = I_y\omega_y, \quad h_z = I_z\omega_z.$$

198. Vector Representation of Angular Momentum. — Similar to the representation of angular velocity by a vector, angular momentum about any line may be represented by a vector along the line. Since in the special case we shall study the angular momentum about any axis used is proportional to the angular velocity about that axis, it is clear that the angular momentum may properly be treated as a vector in the way defined. The signs of the angular momenta will be the same as those of the corresponding angular velocities, and the vectors will be laid off in the same way as for angular velocities.

199. Kinetic Energy of a Body with One Fixed Point. — The kinetic energy of a body at any instant is the sum of the kinetic energy of its particles, or

$$\text{K. E.} = \tfrac{1}{2}\int v^2 dm = \tfrac{1}{2}\int (v_x^2 + v_y^2 + v_z^2)dm.$$

In Art. 197 it was proved that

$$v_x = z\omega_y - y\omega_z, \quad v_y = x\omega_z - z\omega_x, \quad v_z = y\omega_x - x\omega_y.$$

The substitution of these values in the expression for kinetic energy gives

$$\text{K. E.} = \tfrac{1}{2}\int [(z^2 + y^2)\omega_x^2 + (x^2 + z^2)\omega_y^2 + (y^2 + x^2)\omega_z^2]dm$$

In the same way the x-component of the velocity due to rotation about OY is $z\omega_y$.

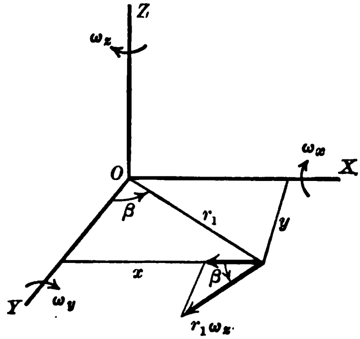


FIG. 274

$$\therefore \frac{dx}{dt} = z\omega_y - y\omega_z.$$

Similarly $$\frac{dy}{dt} = x\omega_z - z\omega_x,$$

and $$\frac{dz}{dt} = y\omega_x - x\omega_y.$$

Substituting these values in the expressions for h_x, we obtain

$$h_x = \int [y(y\omega_x - x\omega_y) - z(x\omega_z - z\omega_x)]dm$$

$$= \int (y^2 + z^2)\omega_x dm - \int xy\omega_y dm - \int xz\omega_z dm,$$

or $$h_x = I_x\omega_x - \omega_y \int xydm - \omega_z \int xzdm.$$

Similarly, $$h_y = I_y\omega_y - \omega_z \int yzdm - \omega_x \int yxdm,$$

and $$h_z = I_z\omega_z - \omega_x \int zxdm - \omega_y \int zydm.$$

The only body whose motion we shall study in this book is a uniform body in the form of a solid of revolution. If one of the axes be taken as the axis of the body, then all of the products of inertia vanish and the expressions for the angular momenta become

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$$\text{K. E.} = \tfrac{1}{2}\int v^2 dm = \tfrac{1}{2}\int (v_x^2 + v_y^2 + v_z^2)dm.$$

In Art. 197 it was proved that

$$v_x = z\omega_y - y\omega_z, \quad v_y = x\omega_z - z\omega_x, \quad v_z = y\omega_x - x\omega_y.$$

The substitution of these values in the expression for kinetic energy gives

$$\text{K. E.} = \tfrac{1}{2}\int [(z^2 + y^2)\omega_x^2 + (x^2 + z^2)\omega_y^2 + (y^2 + x^2)\omega_z^2]dm$$

$$-\int [zy\omega_y\omega_z + xz\omega_x\omega_z + xy\omega_x\omega_y]dm,$$

or,
$$\text{K. E.} = \tfrac{1}{2} I_x\omega_x^2 + \tfrac{1}{2} I_y\omega_y^2 + \tfrac{1}{2} I_z\omega_z^2$$
$$- \omega_y\omega_z\int zydm - \omega_x\omega_z\int xzdm - \omega_x\omega_y\int xydm.$$

For the special case of the body of revolution with one of the coördinate axes coinciding with the axis of the body, or for any body when referred to principal axes,

$$\text{K. E.} = \tfrac{1}{2} I_x\omega_x^2 + \tfrac{1}{2} I_y\omega_y^2 + \tfrac{1}{2} I_z\omega_z^2.$$

200. Vector Rate of Change Due to Rotation.—Let h be a vector with point of application at O. Suppose the vector is changing with the time in magnitude and direction but keeps the point of application the same. Let the vector change in length from h to $h + \Delta h$ in time Δt and in that time turn through the angle $\Delta\theta$. The rate of change in the original direction of the vector is then

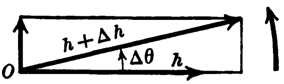


FIG. 275

$$\underset{\Delta t \doteq 0}{\text{Limit}} \frac{(h + \Delta h)\cos\Delta\theta - h}{\Delta t}$$

and the rate of change perpendicular to the original direction of the vector is

$$\underset{\Delta t \doteq 0}{\text{Limit}} \frac{(h + \Delta h)\sin\Delta\theta}{\Delta t}.$$

The first of these quantities may be written

$$\underset{\Delta t \doteq 0}{\text{Limit}} \left[\frac{-h(1 - \cos\Delta\theta)}{\Delta t} + \frac{\Delta h}{\Delta t}\cos\Delta\theta\right]$$

$$= \text{Lim}\left[\frac{-h \cdot 2 \cdot \sin^2 \frac{\Delta\theta}{2}}{\Delta t}\right] + \frac{dh}{dt}$$

$$= \text{Lim}\left[-h \cdot \frac{\sin \frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}} \sin \frac{\Delta\theta}{2} \cdot \frac{\Delta\theta}{\Delta t}\right] + \frac{dh}{dt}$$

$$= -h \cdot 1 \cdot 0 \frac{d\theta}{dt} + \frac{dh}{dt}$$

$$= \frac{dh}{dt}.$$

The second is

$$\underset{\Delta t \doteq 0}{\text{Lim}} (h + \Delta h) \frac{\sin \Delta\theta}{\Delta\theta} \frac{\Delta\theta}{\Delta t}$$

$$= h \cdot 1 \cdot \frac{d\theta}{dt}$$

$$= h\omega,$$

where ω is the angular velocity of rotation of the vector. Hence the rate of change in the direction of the vector is

$$\frac{dh}{dt},$$

and the rate of change at right angles to the vector is

$$h\omega.$$

201. Rate of Change of Angular Momentum of a Body Due to Rotating Axes. — Let the components of the angular momentum of a body with fixed point O with respect to axes OX, OY, OZ be respectively h_1, h_2, h_3, and let this frame of axes coincide at a given instant with a fixed frame OX_1,

OY_1, OZ_1, but be rotating about the fixed frame with angular velocities ω_1, ω_2, ω_3.

Lay off the vectors h_1, h_2, h_3 on the moving axes (Fig. 276). Since the moving axes at the given instant coincide with the fixed axes, the components of the momentum are the same for the fixed and moving axes at that instant. The rate of change of angular momentum along the fixed axes is then given by the method of the preceding article. Thus along OX_1 the rate of change is $\frac{dh_1}{dt}$, due to the change in h_1. This is increased by $h_3\omega_2$ by the rotation of h_3 about the y-axis and by $-h_2\omega_3$ by the rotation of h_2 about the z-axis. Thus the rate of change of angular momentum about the fixed x-axis is

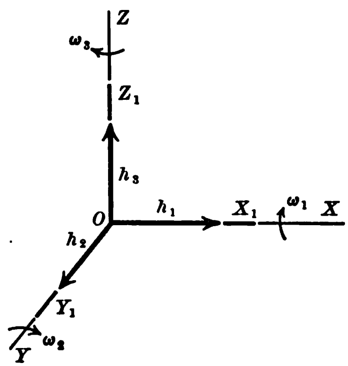


FIG. 276

$$\frac{dh_1}{dt} - h_2\omega_3 + h_3\omega_2.$$

Similarly about the fixed y-axis the rate of change is

$$\frac{dh_2}{dt} - h_3\omega_1 + h_1\omega_3,$$

and about the fixed z-axis the rate of change is

$$\frac{dh_3}{dt} - h_1\omega_2 + h_2\omega_1.$$

Since the moment of impressed forces about any axis is equal to the rate of change of the angular momentum about that axis, we may write

$$T_x = \frac{dh_1}{dt} - h_2\omega_3 + h_3\omega_2,$$

$$T_y = \frac{dh_2}{dt} - h_3\omega_1 + h_1\omega_3,$$

$$T_z = \frac{dh_3}{dt} - h_1\omega_2 + h_2\omega_1.$$

202. Motion of the Center of Gravity of Any Body. — Let dm be the mass of any element of a body and (x, y, z) the coördinates of dm (Fig. 277). The effective forces acting on dm are respectively

$$a_x dm,\ a_y dm, \text{ and } a_z dm.$$

Equating the components of the impressed and effective forces acting on the body, parallel to the x-axis, we have

$$\Sigma X_i = \Sigma a_x dm. \qquad (1)$$

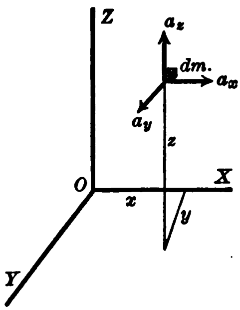


FIG. 277

Now if $\bar{x}$ is the abscissa of the center of gravity of the body,

$$M\bar{x} = \Sigma x dm.$$

Differentiating,

$$M\frac{d\bar{x}}{dt} = \Sigma\frac{dx}{dt}dm,$$

and

$$M\frac{d^2\bar{x}}{dt^2} = \Sigma\frac{d^2x}{dt^2}dm = \Sigma a_x dm.$$

Therefore, if $\bar{a}_x$ is the x-component of the acceleration of the center of gravity of the body, equation (1) may be written

$$\Sigma X_i = M\bar{a}_x.$$

In like manner,

$$\Sigma Y_i = M\bar{a}_y.$$
$$\Sigma Z_i = M\bar{a}_z.$$

Hence, *for any motion of a body the acceleration of the center of gravity is the same as if the whole mass were concentrated there and all the impressed forces were applied there parallel to their original directions.*

203. Gyroscope. Simplest Case. — Consider a body in the form of a solid of revolution turning with uniform angular velocity, ω, about its geometric axis, which is horizontal, while at the same time this axis, passing always through a fixed point, turns with uniform angular velocity, Ω, about the vertical (Fig. 278). The problem is to determine the forces necessary to maintain this motion.

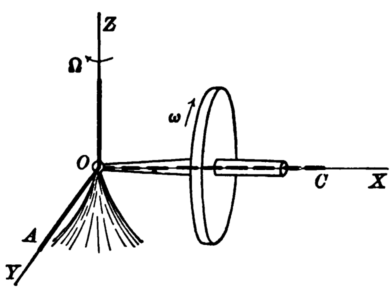


FIG. 278

Let OC be the axis of the body, moving with the body. Choose fixed axes OX, OY, OZ, so that the x-axis coincides with OC at the given instant.

Let OA be the horizontal axis perpendicular to OC. At the given instant OA coincides with OY.

Denote the moment of inertia of the body about OC by C, and that about OA or OZ by A. Then from Art. 197 the angular momenta about OC, OA, and OZ are respectively

$$h_1 = C\omega, \quad h_2 = 0, \quad h_3 = A\Omega.$$

These values of h_1, h_2, and h_3, while they are the values of the angular momenta about the fixed axes OX, OY, OZ, at the given instant, are not values which hold in general, and cannot therefore be differentiated to obtain the torques about these axes. The method of Art. 201 applies here where

$$\omega_1 = 0, \quad \omega_2 = 0, \quad \omega_3 = \Omega.$$

We may write then

$$T_x = \frac{dh_1}{dt} - h_2\omega_3 + h_3\omega_2 = 0, \tag{1}$$

$$T_y = \frac{dh_2}{dt} - h_3\omega_1 + h_1\omega_3 = C\omega\Omega, \tag{2}$$

$$T_z = \frac{dh_3}{dt} - h_1\omega_2 + h_2\omega_1 = 0. \tag{3}$$

For the motion of the center of gravity of the body, since the center of gravity is moving uniformly in a horizontal circle and is at the given instant on the x-axis, the acceleration is directed toward the point of support along the x-axis. Hence

$$a_x = -b\Omega^2, \quad a_y = 0, \quad a_z = 0,$$

where b is the distance from the point of support to the center of gravity.

$$\therefore \Sigma X_i = -Mb\Omega^2, \tag{4}$$

$$\Sigma Y_i = 0, \tag{5}$$

$$\Sigma Z_i = 0. \qquad \text{(Art. 202)} \tag{6}$$

The impressed forces necessary to satisfy equations (1) to (6) may, under certain conditions, be only the weight of the body and the reaction of the support. If these are the only impressed forces, equation (2) becomes

$$Wb = C\omega\Omega,$$

where W is the weight of the rotating body. The remaining equations are then satisfied, so that the necessary and sufficient condition that the body have the motion described is

$$C\omega\Omega = Wb.$$

The motion of the body around the vertical axis is called *precession*. When the angular velocity around the vertical axis is constant, the precession is said to be *steady*.

Problem 454. If the rotating body is a thin disk 2 ft. in diameter fastened to a rod of negligible weight which passes through the center of the disk perpendicular to its faces, the center of the disk being 6 in. from the point of support, what will be the rate of precession in a horizontal plane when the angular velocity of rotation of the disk about its axis is 300 r. p. m.? *Ans.* 9.79 r. p. m.

Problem 455. How would the rate of precession vary with the radius of gyration of the disk if the angular velocity ω be kept unchanged?

204. The Gyroscopic Couple. — Equation (2), Art. 203 shows that when a body of revolution is rotating uniformly about its axis and this axis is rotating uniformly about an axis at right angles to it, there is acting on the body at right angles to the other two axes a couple whose value is the product of the moment of inertia of the body about its axis of revolution and the two angular velocities. This

couple is called the *gyroscopic couple.* In the case of the body discussed in Art. 203, the gyroscopic couple is $C\omega\Omega$.

Since action and reaction are equal and opposite, the rotating body will resist the turning of its plane of rotation with a couple equal and opposite to the gyroscopic couple. This finds application in the effect of rapidly rotating wheels of cars, automobiles, etc., when turning around curves. In Fig. 278 if the disk represents a wheel of a car rolling in the direction indicated by ω and at the same time turning about OZ, the rotation about OZ would be in the opposite direction; *i.e.* Ω would be negative in the formula $T_y = C\omega\Omega$. The gyroscopic couple acting on the wheel would then be in the opposite direction, or about OY in the direction from OX to OZ. The couple that the wheel would exert, tending to upset the car, would then be about OY in the direction from OZ to OX.

Problem 456. A ship carries a cast-iron flywheel whose rim is 6 ft. outside diameter, 4 in. thick, and 18 in. wide. When it is making 3 revolutions per second, its axis is turned about an axis through the plane of the wheel with unit angular velocity. Find the moment of the couple that tends to turn the wheel about an axis perpendicular to these two axes.

Problem 457. A solid cast-iron disk 3 ft. in diameter and 3 in. thick revolves about its axis, making 3000 revolutions per minute. At the same time it is made to turn about an axis in its plane at the rate of 2 revolutions per minute. Find the magnitude of the couple tending to rotate the disk about an axis perpendicular to these two axes.

Problem 458. A locomotive is going at the rate of 40 mi. per hour around a curve of 600 ft. radius. The diameter of the drivers is 80 in., and a pair of drivers and axle have a moment of inertia

about an axis midway between the wheels and perpendicular to the axle of 3000. What is the magnitude of the couple introduced by the precessional motion of this pair of wheels? Give the direction in which it acts. Does it tend to make the locomotive tip inward or outward?

Problem 459. A car pulled by the locomotive in the preceding problem has four pairs of wheels. The moment of inertia of each pair of wheels and their connecting axle, with respect to an axis midway between the wheels and perpendicular to the axle, is 320. What is the magnitude of the precessional couple acting upon the whole car?

Problem 460. The flywheel of an engine on board a ship makes 300 revolutions per minute. The rim has the following dimensions: outside radius 4 ft., inside radius $3\frac{1}{2}$ ft., width 12 in. The ship rolls with an angular velocity of $\frac{1}{2}$ a radian per second; find the torque acting on the ship due to the gyrostatic action of the flywheel.

205. Car on Single Rail. — An interesting application of the gyroscope has been made recently in England. A car is run upon a single rail, and is held upright by means of rapidly rotating flywheels. Each car contains two of these wheels rotating in opposite directions, at the rate of 8000 revolutions per minute.

Any tendency of the car to tip over, either when running or standing at the station, is righted by the gyroscopic action of the flywheels. The experimental car was so successful in operation that it maintained itself in an upright position even when loaded eccentrically. The action of the flywheels is such as to place the center of gravity of the car and load directly over the rails. See *Engineering*, June 7, 1907.

Another practical application of the gyroscopic couple is to be found in Schlick's "stabilizer" for steadying ships.

206. Gyroscope. Inclined Axis. — Let the angular velocity of the body relative to its own axis, OC, be ω at any instant, the angular velocity with which the plane containing the axis turns about the vertical be Ω, and the angle which the axis of the body makes with the vertical be θ. The component about OC of the total angular velocity of the body is then $\omega +$ the component of Ω about OC or $\omega + \Omega \cos \theta$.

The angular velocity ω is called the *velocity of spin*.

Let $\dfrac{d\theta}{dt} = \omega'$. Then ω' is the angular velocity of the axis of the body in the vertical plane containing the axis.

Choose fixed axes so that at the instant in question the xz-plane contains the axis of the body, the x-axis being horizontal (Fig. 279). Let OA, OB, OC be a set of moving axes, of which OC is the axis of the rotating body, OB is horizontal, and OA is at right angles to OC and OB. Let OA_1, OB_1, OC_1 be fixed axes which at the given instant coincide with OA, OB, OC respectively. The frame of moving axes $OABC$ then has angular velocities about the fixed axes OA_1, OB_1, OC_1 respectively

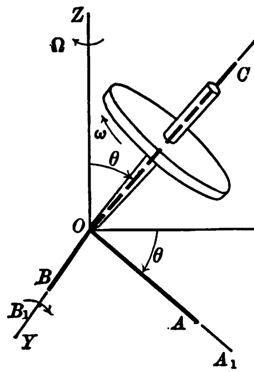


FIG. 279

$$\omega_1 = -\Omega \sin \theta, \quad \omega_2 = \omega', \quad \omega_3 = \Omega \cos \theta.$$

The angular velocity Ω may be resolved into components

about OA, OB, OC respectively

$$-\Omega\sin\theta,\quad 0,\quad \Omega\cos\theta.$$

The angular velocities of the body about the axes OA, OB, OC respectively are therefore

$$-\Omega\sin\theta,\quad \omega',\quad \text{and } \omega+\Omega\cos\theta.$$

If the moments of inertia of the body about the axes OA, OB, OC are respectively A, B, C, or, because of symmetry, A, A, C, the angular momenta of the body about these axes are respectively

$$h_1 = -A\Omega\sin\theta,\quad h_2 = A\omega',\quad h_3 = C(\omega+\Omega\cos\theta).$$

The discussion will be divided into two cases: (*a*) steady precession, (*b*) unsteady precession.

(*a*) *Steady Precession.* Suppose the angle θ and the angular velocities ω and Ω are all constant. For this case $\omega' = 0$ and

$$\begin{aligned} h_1 &= -A\Omega\sin\theta, & h_2 &= 0, & h_3 &= C(\omega+\Omega\cos\theta),\\ \omega_1 &= -\Omega\sin\theta, & \omega_2 &= 0, & \omega_3 &= \Omega\cos\theta. \end{aligned}$$

Also, equating the torques and the rates of change of the angular momenta about the axes OA_1, OB_1, OC_1, we have

$$T_a = \frac{dh_1}{dt} - h_2\omega_3 + h_3\omega_2 = 0, \tag{1}$$

$$\begin{aligned} T_b &= \frac{dh_2}{dt} - h_3\omega_1 + h_1\omega_3 \\ &= C\Omega\sin\theta(\omega+\Omega\cos\theta) - A\Omega^2\sin\theta\cos\theta, \end{aligned} \tag{2}$$

$$T_c = \frac{dh_3}{dt} - h_1\omega_2 + h_2\omega_1 = 0. \tag{3}$$

It is possible for this motion to take place under the action of the weight and the reaction of the support as the only impressed forces for certain related values of ω, Ω, and the constants of the body. For if only the weight and the support act on the body, then, since the center of gravity moves as if all the forces were concentrated there, the only additional conditions to be satisfied are

$$X_r = -Mb\sin\theta\Omega^2, \tag{4}$$
$$Y_r = 0, \tag{5}$$
$$Z_r = W, \tag{6}$$

where X_r, Y_r, Z_r are the components of the reaction of the support along the fixed axes at the given instant, and b is the distance from the point of support to the center of gravity of the body.

With only the weight and the reactions of the support as impressed forces, the torques T_a and T_c are both zero and $T_b = Wb\sin\theta$. All of the conditions of motion would then be satisfied if, from equation (2),

$$Wb\sin\theta = C\Omega\sin\theta(\omega + \Omega\cos\theta) - A\Omega^2\sin\theta\cos\theta,$$
or
$$(A - C)\cos\theta\Omega^2 - C\omega\Omega + Wb = 0.$$

The values of Ω that satisfy this equation are

$$\Omega = \frac{C\omega \pm \sqrt{C^2\omega^2 - 4\,Wb(A - C)\cos\theta}}{2(A - C)\cos\theta}.$$

There are therefore two solutions, one solution, or no solution according as

$$C^2\omega^2 \gtreqless 4\,Wb(A - C)\cos\theta.$$

Problem 461. Find the value, or values, of Ω for steady precession with axis inclined 75° from the vertical of the disk of Problem 454, the thickness of the disk being negligible.

What is the least value of ω that would satisfy the conditions for the motion of this disk when $\theta = 75°$?

Problem 462. A conical top is made of wood and is spinning about its axis with a velocity of 20 revolutions per second. The cone has a base of 2 in. and a height of 2 in., and spins on the apex. While spinning steadily with its axis vertical (sleeping), it is disturbed by a blow so that its axis is inclined at an angle of 30° with the vertical. Find the velocity of precession and the torque that tends to keep the top from falling.

(*b*) *Unsteady Precession.* Assume θ, ω, and Ω to be variables. The equations then become

$$\omega_1 = -\Omega \sin\theta, \qquad \omega_2 = \omega', \qquad \omega_3 = \Omega\cos\theta,$$

$$h_1 = -A\Omega\sin\theta, \qquad h_2 = A\omega', \qquad h_3 = C(\omega + \Omega\cos\theta),$$

$$T_a = -A\left(\Omega\cos\theta\cdot\omega' + \sin\theta\frac{d\Omega}{dt}\right) - A\omega'\Omega\cos\theta + C\omega'(\omega + \Omega\cos\theta), \quad (1')$$

$$T_b = A\frac{d\omega'}{dt} + C\Omega\sin\theta(\omega + \Omega\cos\theta) - A\Omega^2\sin\theta\cos\theta, \quad (2')$$

$$T_c = C\frac{d}{dt}(\omega + \Omega\cos\theta). \quad (3')$$

Again suppose the weight and the reaction of the support to be the only impressed forces. Then $T_a = 0$, $T_b = Wb\sin\theta$, $T_c = 0$. Equation (3′) then becomes

$$\frac{d}{dt}(\omega + \Omega\cos\theta) = 0.$$

Therefore

$$\omega + \Omega\cos\theta = \text{a constant.}$$

For the sake of simplicity suppose the body to have been placed with its geometric axis at rest making an angle θ_0 with the vertical, the body then given an angular velocity ω_0 about its axis and released. Then the value of $\omega + \Omega \cos \theta$ at the start was ω_0. Therefore throughout the motion

$$\omega + \Omega \cos \theta = \omega_0.$$

Substituting this value in equation (1′) where $T_a = 0$, and replacing ω' by $\frac{d\theta}{dt}$, it becomes

$$2 A\Omega \cos \theta d\theta + A \sin \theta d\Omega - C\omega_0 d\theta = 0.$$

If this equation be multiplied through by $\sin \theta$, it can then be integrated, for it becomes

$$A(2\,\Omega \sin \theta \cos \theta d\theta + \sin^2 \theta d\Omega) - C\omega_0 \sin \theta d\theta = 0,$$

in which the quantity inclosed in the parenthesis is $d(\Omega \sin^2 \theta)$.

Integrating,

$$A\Omega \sin^2 \theta + C\omega_0 \cos \theta = \text{constant}.$$

At the beginning of motion $\Omega = 0$, and $\theta = \theta_0$. Therefore

$$A\Omega \sin^2 \theta = C\omega_0 (\cos \theta_0 - \cos \theta). \qquad (5')$$

This equation expresses Ω in terms of θ.

Instead of using equation (2′) a simpler equation is obtained by considerations of work and energy. The work done from the beginning of motion to the instant under consideration is that done by gravity, *i.e.*

$$Wb(\cos \theta_0 - \cos \theta).$$

The gain in kinetic energy in this time is

$$\{\tfrac{1}{2}A(-\Omega\sin\theta)^2+\tfrac{1}{2}A\omega'^2+\tfrac{1}{2}C\omega_0^2\}-\tfrac{1}{2}C\omega_0^2. \quad \text{(Art. 199)}$$

Therefore

$$\tfrac{1}{2}A(\Omega^2\sin^2\theta+\omega'^2)=Wb\,(\cos\theta_0-\cos\theta),$$

or
$$\omega'^2=\frac{2\,Wb}{A}(\cos\theta_0-\cos\theta)-\Omega^2\sin^2\theta.$$

Replacing $\cos\theta_0-\cos\theta$ by $\dfrac{A\Omega\sin^2\theta}{C\omega_0}$ from equation (5′),

$$\omega'^2=\Omega\left(\frac{2\,Wb}{C\omega_0}-\Omega\right)\sin^2\theta. \tag{6′}$$

The factors Ω and $\dfrac{2\,Wb}{C\omega_0}-\Omega$ must both be of the same sign since ω'^2 must be positive. Assuming ω_0 to be positive, it follows that Ω is positive, for if Ω were negative, the second factor would be positive and their product would be negative.

Also the values of Ω that make ω' zero are 0 and $\dfrac{2\,Wb}{C\omega_0}$.

From equation (5′)

$$\Omega=\frac{C\omega_0}{A}\cdot\frac{\cos\theta_0-\cos\theta}{\sin^2\theta},$$

from which it can be shown that

$$\frac{d\Omega}{d\theta}=\frac{C\omega_0}{A\sin^3\theta}\left[(\cos\theta_0-\cos\theta)^2+\sin^2\theta_0\right].$$

Since $\sin\theta$ is positive, $\dfrac{d\Omega}{d\theta}$ is positive; *i.e.* Ω increases as θ increases and decreases as θ decreases. It follows then that θ and Ω both increase until $\Omega=\dfrac{2\,Wb}{C\omega_0}$. ω' is then 0, θ

changes from increasing to decreasing and continues to decrease until $\Omega = 0$ and $\theta = \theta_0$ (Equations (6′) and (5′)).

The axis of the body therefore alternately falls and rises between the values $\theta = \theta_0$ and $\theta = \theta_1$, where θ_1 is the value obtained by putting $\Omega = \dfrac{2\,Wb}{C\omega_0}$ in equation (5′). As the axis of the body falls and rises, the value of Ω increases from 0 when $\theta = \theta_0$ to $\dfrac{2\,Wb}{C\omega_0}$ when $\theta = \theta_1$, decreasing again to 0 when $\theta = \theta_0$.

The condition for this motion is that the value of θ obtained from equations (5′) shall be real.

Problem 463. Using the body of Problem 454, making $\omega_0 = 600$ r. p. m. and $\theta_0 = 30°$, find the range of values for θ and Ω.

Problem 464. For what least value of ω_0 would the body of the preceding problem have the motion described in this article, the other conditions being the same?

CHAPTER XV

IMPACT

207. Definitions. — When two bodies collide, they are said to be subject to *impact.*

When the velocity of the striking surfaces is in the direction of the normal to those surfaces, the impact is said to be *direct.* Otherwise it is *oblique.*

When the normal to the surfaces at the point of contact (or center of forces exerted by one body on the other) passes through both centers of gravity of the bodies, the impact is said to be *central.*

The phenomena of impact may be best studied by considering the two bodies somewhat elastic. Suppose for simplicity that they are two spheres, M_1 and M_2 (Fig. 280), and that they are moving with velocities v_1 and v_2 and that the impact is central. In Fig. 280 (*a*) they are shown at the instant when contact first takes place, and in Fig. 280 (*b*) they are shown some time after first contact when each has been deformed somewhat by the pressure of the other. The dotted lines indicate the original spherical form and the full lines the assumed form of the deformed spheres. When the spheres first touch, the pressure P between them is zero, but as each one compresses the other, the pressure P increases until it becomes a maximum. The compression of the spheres is indicated in the figure by d_1 and d_2. We

shall designate the time during which the bodies are being compressed as the *period of deformation.*

After the compression has reached its maximum value the bodies, if they be partially elastic, begin to separate and to regain their original form. The common pressure P decreases and becomes zero, if the bodies are sufficiently elastic so that they finally separate. We shall designate this period of separation as the *period of restitution*, and the velocities after separation as v_1' and v_2'.

If the bodies are entirely *inelastic*, there will be no restitution. They will, in that case, remain in contact just as they are when the pressure between them is a maximum and will move on with a common velocity V.

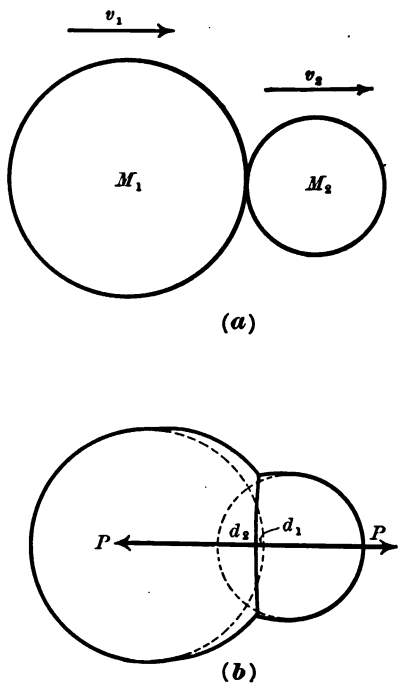


Fig. 280

In what follows velocities and accelerations toward the right will be called positive and those toward the left negative.

208. Direct Central Impact. — When the bodies meet in direct central impact, separation will take place along the

line joining the centers of gravity. Let T be the time from the first contact up to the time of maximum pressure, that is, the *time of deformation*, and T_1 the time from first contact up to the time of separation. Then $T_1 - T$ represents the *time of restitution*. We have, from Art. 103, $dv = a \cdot dt$. Considering the motion of M_1 during the period of deformation, we have $dv = adt$ and

$$a = -\frac{P}{M_1},$$

so that
$$\int_{v_1}^{V} dv = -\frac{1}{M_1}\int_0^T Pdt,$$

or
$$M_1(V - v_1) = -\int_0^T Pdt,$$

where V is the common velocity of the centers of gravity of the bodies at the end of the period of deformation.

In a similar way,

$$M_2(V - v_2) = \int_0^T Pdt.$$

The two integrals $\int_0^T Pdt$ on the right-hand side of the preceding equations cannot be determined since we do not know in general how the pressure P varies with the time; we do know, however, that they are equal term for term, so that we may eliminate them. We have, then,

$$-M_1(V - v_1) = M_2(V - v_2),$$

or
$$(M_1 + M_2)V = M_1v_1 + M_2v_2.$$

If the impact is not too severe, elastic or partially elastic bodies tend to regain their original shape after the deformation has reached a maximum and finally separate if they

possess sufficient elasticity. Using the notation of Art. 207, we have, for the period of restitution, if R is the force of restitution,

$$M_1\int_V^{v_1'} dv = -\int_T^{T_1} Rdt,$$

$$M_2\int_V^{v_2'} dv = \int_T^{T_1} Rdt,$$

so that $$M_1(v_1' - V) = -\int_T^{T_1} Rdt,$$

and $$M_2(v_2' - V) = \int_T^{T_1} Rdt,$$

from which $$(M_1 + M_2)V = M_1v_1' + M_2v_2'.$$

The value of the integral $\int Pdt$ during deformation will not in general be the same as its value during restitution. Call the ratio of $\int_T^{T_1} Rdt$ to $\int_0^T Pdt$, e. This value, which is called the *coefficient of restitution*, is constant for given materials. It is unity for perfectly elastic substances, zero for non-elastic substances, and some intermediate value for the imperfectly elastic materials with which the engineer is usually concerned. The following values of e have been determined: for steel on steel, $e = .55$; for cast iron on cast iron, $e = 1$, nearly ; for wood, $e = 0$, nearly.

From the above definition of e, it is seen at once that we may write

$$e = \frac{v_1' - V}{V - v_1}, \text{ and } e = \frac{v_2' - V}{V - v_2},$$

from which it follows that

$$v_1' = V(1 + e) - ev_1,$$

$$v_2' = V(1+e) - ev_2,$$

where $$V = \frac{M_1 v_1 + M_2 v_2}{M_1 + M_2}.$$

From the above equations it is seen that

$$v_1' - v_2' = -e(v_1 - v_2).$$

209. Momentum and Kinetic Energy in Impact. — When a constant force acts upon a body for a certain time, the change in the velocity of the body is directly proportional to the force and to the time and inversely proportional to the mass of the body.

The product of the force, F, and the time, t, is called the *impulse of the force*, and the product of the mass and velocity is called the *momentum* of the body. (The body is supposed to have no rotation.)

From Newton's law, $F = Ma$, it follows that

$$Ft = M(v_1 - v_0),$$

or, *when a body is acted upon by a constant force, the impulse of the force is equal to the change in momentum of the body.*

If the force is not constant, and is represented at any time by P, then $\int_0^t P dt$ is the impulse of the force for the time from 0 to t, and from the equations of the preceding article,

$$\int_0^T P dt = -M_1(V - v_1),$$

$$\int_0^T P dt = M_2(V - v_2),$$

it follows that the impulse of the force is equal to the change of the momentum of the body on which it acts, no other force being assumed to act on the body.

From the equations of the preceding article,

$$(M_1 + M_2)V = M_1v_1 + M_2v_2,$$

and

$$(M_1 + M_2)V = M_1v_1' + M_2v_2',$$

it follows that there is no change in the sum of the momenta of the bodies in impact, either during the period of deformation or the period of restitution. This is otherwise evident from the fact that the forces acting on the bodies are equal and opposite, and hence what one body gains in momentum the other loses.

With the kinetic energy it is different. During the period of deformation the loss in kinetic energy is

$$\begin{aligned} E_1 &= \tfrac{1}{2}M_1v_1^2 + \tfrac{1}{2}M_2v_2^2 - \tfrac{1}{2}(M_1 + M_2)V^2 \\ &= \tfrac{1}{2}\left[M_1v_1^2 + M_2v_2^2 - \frac{(M_1v_1 + M_2v_2)^2}{M_1 + M_2}\right] \\ &= \frac{M_1M_2}{2(M_1 + M_2)}(v_1 - v_2)^2. \end{aligned}$$

During the period of restitution there is a gain in kinetic energy,

$$\begin{aligned} E_2 &= \frac{M_1M_2}{2(M_1 + M_2)}(v_1' - v_2')^2 \\ &= \frac{e^2M_1M_2}{2(M_1 + M_2)}(v_1 - v_2)^2, \end{aligned}$$

since

$$v_1' - v_2' = -e(v_1 - v_2).$$

The total loss in kinetic energy during the impact is therefore

$$E_1 - E_2 = \frac{M_1M_2(1 - e^2)(v_1 - v_2)^2}{2(M_1 + M_2)}.$$

If the bodies are inelastic, $e = 0$, and the bodies go on with the common velocity V. The loss in kinetic energy in this case is

$$\frac{1}{2}\frac{M_1 M_2}{M_1 + M_2}(v_1 - v_2)^2.$$

If the bodies are perfectly elastic, $e = 1$, and the loss in kinetic energy is zero.

Problem 465. A lead sphere whose radius is 2 in. strikes a large mass of cast iron after falling freely from rest through a distance of 100 ft. What is its final velocity? What is the loss of kinetic energy? Assume $e = 0$.

Problem 466. A 10-lb. lead sphere is at rest when it is acted upon by another lead sphere, whose radius is 3 in., in direct central impact. The velocity of the latter sphere is 20 ft. per second. What is the common velocity of the two spheres and what is the loss of kinetic energy due to impact?

Problem 467. Prove that if two inelastic bodies, moving in opposite directions with speeds inversely proportional to their masses, collide, both will be reduced to rest.

Problem 468. Prove that if two perfectly elastic bodies with equal masses collide, each mass will have after impact the velocity that the other had before. If one of the bodies is at rest and the other strikes it, what will happen?

Problem 469. Figure 281 represents a set of perfectly elastic balls of equal mass and size in contact in a straight line. Another ball, of the same mass and material, moving in the direction of the line of balls, strikes one end of the line. Prove that the moving ball will be brought to rest where it strikes and the last ball at the other end of the line will take its velocity, all other balls remaining at rest. Prove also that if two balls, moving together,

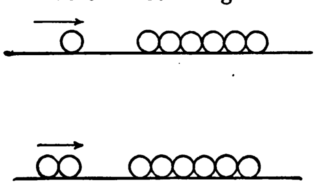


Fig. 281

strike the line, the two balls at the other end of the line will take the velocity of the striking balls and the latter will be brought to rest.

Problem 470. Prove that if a ball of mass M_1 fall from a height H upon a very large mass with a horizontal surface and rebounds to a height h, the coefficient of restitution of the materials used is

$$e = \sqrt{\frac{h}{H}}.$$

(Regard M_2 as infinite and $V = 0$.)

Problem 471. A bullet weighing 1 oz. is fired horizontally into a box of sand (inelastic) weighing 20 lb., and remains imbedded in the sand. The box is suspended by a string attached to a fixed point 4 ft. from the center of the box of sand. The impact of the bullet causes the box to swing aside through an angle of 42°. Find (*a*) the velocity of impact of the bullet, (*b*) the kinetic energy lost in impact, (*c*) the greatest tension in the string.

Problem 472. Assuming inelastic impact, show that if a pile driver weighing G_1 lb. falls h ft. and drives a pile weighing G_2 lb. s ft. into the ground against a constant resistance R,

$$R = \frac{G_1^2}{G_1 + G_2}\frac{h}{s}.$$

(HINT. Equate work done against resistance plus kinetic energy lost in impact to the work done by gravity on the falling weight.)

Problem 473. A wooden pile weighing 1500 lb. is driven by a steel hammer, of weight 2000 lb., falling 20 ft. The penetration at the last blow is observed to be $\frac{1}{2}$ in. What is the resistance offered by the earth to the pile? With a safety factor 6, what load would the pile carry?

210. Elasticity of Material. — All materials of engineering are imperfectly elastic. Some, however, show almost perfect elasticity for stresses that are rather low. This has been expressed by saying that all materials have a

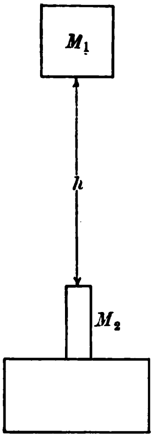
limit (elastic limit) beyond which if the stress be increased the material will be imperfectly elastic. Within the *limit of elasticity*, *stress* is proportional to the *deformation* produced. Let P be the total stress in tension or compression, f the stress, in pounds per square inch of cross section, d be the deformation caused by P and λ the deformation per inch of length. Within the limit of elasticity of the material the ratio $\frac{f}{\lambda}$ is a constant, and since $f=\frac{P}{F}$, and $\lambda=\frac{d}{l}$, when F is the area of cross section and l is the length of the material, it may be written $\frac{Pl}{Fd}$. This constant is called the *modulus of elasticity* of the material; it is usually represented by E, so that

$$E=\frac{f}{\lambda}=\frac{Pl}{Fd}$$

for tension or compression. For steel E has been found to be about 30,000,000 lb. per square inch.

211. Impact Tension and Impact Compression. — Figure 282 (a) represents a mass M_2 subjected to impact from the mass M_1 falling from rest through a height h. The mass M_2 is compressed by the impact. Figure 282 (b) represents the body M_2 as subjected to impact in tension, the mass in this case being a rod having M_1 attached to one end and the other end attached to a crosshead A. The rod, crosshead, and weight fall freely together through the distance h until A strikes the stops at B, when one end of the rod suddenly comes to rest and the mass M_1 causes tension in the rod due to impact.

Suppose v_1 represents the velocity of M_1 when impact occurs and l_2 the length of M_2, whether it be a tension or compression piece.



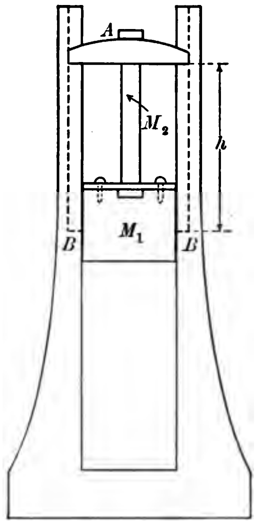


FIG. 282

In case (b) when the weights strike the stops at B the kinetic energy of the falling weights is equal to

$$(G_1 + G_2)h \text{ ft.-lb.},$$

where G_1 and G_2 are the weights of M_1 and M_2 in pounds and h is in feet. This kinetic energy is used up in doing work in compressing the stops and in stretching M_1 and M_2. If M_2 is a rod of small cross section, its length is increased a large amount compared to the change in length of M_1 and the stops against which the crosshead strikes,

and we may without serious error assume that all the kinetic energy is used up in stretching M_2. We may assume too that the stress in M_2 is the same throughout its length.

Let P_m be the maximum stress induced in the rod. The average stress is then $\frac{P_m}{2}$, and if d_m is the total increase in the length of the rod, the work done on the rod is

$$\frac{P_m d_m}{2}.$$

$$\therefore \frac{P_m d_m}{2} = (G_1 + G_2)h.$$

But $$P_m = \frac{F_2 d_m E_2}{l_2}. \qquad \text{(Art. 210)}$$

$$\therefore d_m^2 = \frac{2(G_1 + G_2)hl_2}{F_2 E_2}.$$

or $$d_m = \sqrt{\frac{2(G_1 + G_2)hl_2}{F_2 E_2}}.$$

Here E must be expressed in pounds per square foot and all dimensions in feet if G_1 and G_2 are in pounds and h in feet.

In case (a) the student can easily show that

$$d_m = \sqrt{\frac{2\, G_1 h l_2}{F_2 E_2}}.$$

Problem 474. Derive the formula just given for the case (a).

Problem 475. A weight of 500 lb. falls through a distance of 2 ft. in such a way as to put a 1-in. round steel rod in tension. If the rod is 18 ft. long, what will be the elongation due to the impact? Use $E = 30{,}000{,}000$ lb. per square inch for steel. What maximum unit stress is induced in the rod?

Problem 476. A cylindrical piece of steel 1 in. high and 1 in. in diameter is subjected to compression by a weight of 20 lb. falling through a distance of 1 in. How much will it be compressed?

Problem 477. In the preceding problem what stress (pounds per square inch) was caused in the cylindrical block by the fall of the 20-lb. weight?

Problem 478. The safe stress in structural steel for moving loads is usually taken as 12,500 lb. per square inch. Through what height might a 300-lb. weight fall so as to produce tension in a 1-in. steel round bar, 10 ft. long, without exceeding the safe stress?

Problem 479. Two steel tension rods in a bridge, each 2 sq. in. in cross section and 20 ft. long, carry the effect of the impact of a loaded wagon as one wheel rolls over a stone 1 in. high. The weight on the wheel is 2000 lb. What stress is introduced in the tension rods?

NOTE. For the strength of metals under impact the student is referred to the work of W. K. Hatt, Am. Soc., "Testing Materials," Vol. IV, p. 282.

212. Direct Eccentric Impact. — Let M_1 and M_2 have impact as shown in Fig. 283, in which the centers of gravity of the bodies are moving along parallel lines and the surfaces of contact are perpendicular to the line of motion. This is known as *direct eccentric impact.*

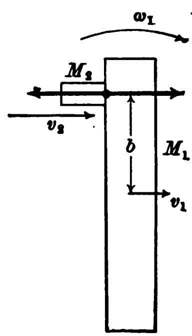


FIG. 283

Let the velocity of M_2 before impact be v_2, the velocity of the center of gravity of M_1 be v_1, and its angular velocity be ω_1.

(*a*) The problem will first be solved on the assumption that M_1 is acted upon by no forces except that of impact.

First period of impact. Let P be the force at any in-

stant of the first period of impact, T the time of the first period, V the velocity of M_2, ω the angular velocity of M_1, and V_1 the velocity of the center of gravity of M_1 at the end of the first period.

During the first period, which lasts for a very short time, the position of the body M_1 may be regarded as unchanged. We then have for the motion of M_2,

$$-\int_0^T Pdt = M_2\int_{v_2}^{V} dv = M_2(V - v_2), \tag{1}$$

and for the motion of M_1

$$\int_0^T Pdt = M_1\int_{v_1}^{V_1} dv = M_1(V_1 - v_1), \tag{2}$$

and
$$bP = I\frac{d\omega}{dt}$$

or
$$b\int_0^T Pdt = M_1k^2\int_{\omega_1}^{\omega} d\omega = M_1k^2(\omega - \omega_1), \tag{3}$$

where k is the radius of gyration of M_1 about a gravity axis perpendicular to the plane of motion. Also, since at the end of the first period the velocity of M_2 is the same as the velocity of the point of impact on M_1,

$$V = V_1 + b\omega. \tag{4}$$

The unknowns in these four equations are

$$\int_0^T Pdt,\ V,\ V_1,\ \omega.$$

The equations are linear in these unknowns and the unknowns can be easily determined from them.

Second period of impact. If R is the force of impact at any time during the second period of impact, T the value of t, v_2' the velocity of M_2, v_1' the velocity of the center of

gravity of M_1, and ω_1' the angular velocity of M_1 at the end of the second period, then

$$-\int_T^{T'} Rdt = M_2\int_V^{v_2'} dv = M_2(v_2' - V), \qquad (5)$$

$$\int_T^{T'} Rdt = M_1\int_V^{v_1'} dv = M_1(v_1' - V_1), \qquad (6)$$

$$b\int_T^{T'} Rdt = M_1k^2\int_\omega^{\omega_1'} d\omega = M_1k^2(\omega_1' - \omega). \qquad (7)$$

Also $$\int_T^{T'} Rdt = e\int_0^T Pdt. \qquad (8)$$

Here again are four linear equations in the four unknowns, $$\int_T^{T'} Rdt,\ v_1',\ v_2',\ \omega_1',$$

which are therefore sufficient to determine the unknowns.

(*b*) *Impact on a body with fixed axis.* Suppose the body rotating about the fixed axis through O (Fig. 284), with angular acceleration α_1 and angular velocity ω_1 at the beginning of impact.

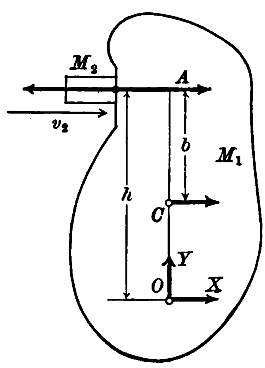


FIG. 284

Choose the origin of coördinates at O and the x-axis coinciding with the direction of motion of the centers of gravity of the bodies.

Let X and Y be the reactions of the supporting axis through O at any time during the impact.

First period of impact. For the first period we have the following equations:

$$-\int_0^T Pdt = M_2(V - v_2), \qquad (1')$$

$$\int_0^T (P + X)dt = M_1(V_1 - v_1), \qquad (2')$$

$$\int_0^T [Pb - X(h - b)]dt = M_1 k^2(\omega - \omega_1). \qquad (3')$$

Also, since O is fixed,

$$V_1 = (h - b)\omega, \qquad (4')$$

$$V = h\omega. \qquad (5')$$

Here are five equations containing five unknowns:

$$\int_0^T Pdt, \quad \int_0^T Xdt, \quad V_1, \quad V, \quad \omega,$$

from which the unknowns may be determined.

Second period of compact. For the second period the following equations hold:

$$-\int_T^{T'} Rdt = M_2(v_2' - V), \qquad (6')$$

$$\int_T^{T'} (R + X)dt = M_1(v_1' - V_1), \qquad (7')$$

$$\int_T^{T'} [Rb - X(h - b)]dt = M_1 k^2(\omega_1' - \omega), \qquad (8')$$

$$v_1' = (h - b)\omega_1', \qquad (9')$$

$$\int_T^{T'} Rdt = e\int_0^T Pdt. \qquad (10')$$

Here again are five equations containing five unknowns:

$$\int_T^{T'} Rdt, \quad \int_T^{T'} Xdt, \quad v_1', \quad v_2', \omega_1'.$$

These equations are sufficient to determine the unknowns, and thus the motion of the bodies is given after impact.

In the above equations k is the radius of gyration of M_1 about a gravity axis perpendicular to the plane of motion.

Problem 480. A uniform steel bar, 4 ft. long, weighing 16 lb., lies at rest on a smooth horizontal plane. It is struck by a steel ball weighing 2 lb., with a velocity of 30 f/s, at right angles to the rod at a point 6 in. from one end. Find the velocity of the ball, the velocity of the center of gravity of the bar, and its angular velocity at the end of the first period, and at the end of the second period of impact, given $e = .55$.

Problem 481. In the preceding problem find the point about which the bar is turning at the end of the first period. Is this point the center of rotation at the end of the second period?

Problem 482. If M_1 is at rest when M_2 strikes with direct eccentric impact, write the equations which determine the motion at the end of the first, and at the end of the second period of impact.

Problem 483. If $e = 0$ between M_1 and M_2 of Art. 212 (*a*), find the kinetic energy lost in impact.

Problem 484. Suppose M_1 to be a rod of steel $\frac{1}{2}$ in. in diameter and 2 ft. long, and suppose M_2 to be a hammer weighing 2 lb. and that its velocity at the time of impact is 20 ft. per second, find V and ω, if the hammer strikes 10 in. from the center of the rod. $e = .55$.

Problem 485. Let M_1 be a square stick of timber $4'' \times 4'' \times 10'$ and let M_2 be a 10-lb. hammer having a velocity at the time of impact of 10 ft. per second. If the impact takes place 4 ft. from the center, find V and ω, given $e = .10$.

213. Center of Percussion and Center of Instantaneous Rotation. — Consider a body at rest to be struck a blow and to be free to move in the plane passing through the center of gravity of the body and the line in which the blow is struck; as, for example, a slab at rest on a smooth horizontal plane when struck a blow in a horizontal plane through the center of gravity of the body.

Let C be the center of gravity of the body and P the force of the blow at any time during the impact (Fig.

285). Then, since for the very brief period in which the blow takes place any change in the position of the body is negligible, we have for the motion of the body at any instant during the blow, if v is the velocity of the center of gravity, and ω the angular velocity,

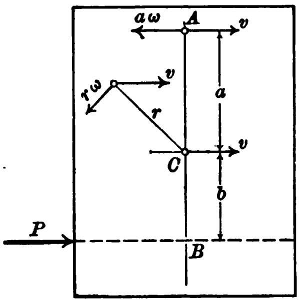


FIG. 285

$$P = M\frac{dv}{dt},$$

$$Pb = Mk^2\frac{d\omega}{dt},$$

where k is the radius of gyration about a gravity axis perpendicular to the plane of motion of the center of gravity of the body.

If t is the time at any instant of the blow, measured from the beginning of the blow, there follow from the above equations,

$$\int_0^t Pdt = M\int_0^v dv = Mv,$$

and
$$b\int_0^t Pdt = Mk^2\int_0^\omega d\omega = Mk^2\omega.$$

By division,
$$\frac{1}{b} = \frac{v}{k^2\omega}.$$

The velocity of any point of the body at the given instant is composed of two components, one the velocity, v, of the center of gravity, and the other, $r\omega$, relative to the center of gravity. There is then one point of the body, or in a plane fixed in the body and moving with it, where these two components just annul each other. This point (A, Fig. 285) must lie on the line through the

center of gravity perpendicular to the line of the blow at a distance a from the center of gravity such that

$$a\omega = v.$$

But $$\frac{1}{b} = \frac{v}{k^2\omega}.$$

By division, $$ab = k^2,$$

from which a is determined when b and k are known.

There is, therefore, an axis for each blow about which the body begins to turn, and this axis remains approximately the same as long as the change in the position of the body is negligible; that is, in general, during the blow.

The axis about which the body begins to turn under the action of the blow is called the ***instantaneous axis of rotation.***

The point, A, in this axis which lies in the plane containing the center of gravity and the line of the blow is called the ***instantaneous center of rotation.***

If a fixed axis passed through the body at A, perpendicular to the plane containing the center of gravity and the line of the blow, the blow would cause the body to turn about this axis without causing any sudden reaction of the axis on the body. The point B, the foot of the perpendicular from the center of gravity to the line of the blow, is known in this case as the ***center of percussion*** of the body corresponding to the center of rotation A.

It should be noted that center of percussion and center of instantaneous rotation are related just as center of suspension and center of oscillation are in the compound pendulum (Art. 182).

Problem 486. A thin rod, 3 ft. long, is struck a blow at 6 in. from one end and at right angles to the rod. Find the instantaneous center of rotation.

Problem 487. A right circular steel cylinder of diameter 6 in. and height 2 ft. is suspended by an axis in a diameter of one end. It is struck a blow with a 5-lb. hammer with a velocity of 20 f/s through the center of percussion corresponding to the given support. Find the velocity of the hammer and the angular velocity of the cylinder at the time of greatest pressure and at the end of the impact, given $e = .55$.

Problem 488. A right circular cone of steel, the radius of whose base is 6 in. and altitude 6 in., is supported as a pendulum by an axis through its vertex parallel to the base. It is struck with a 3-lb. hammer with a velocity of 10 ft. per second, in a line through the center of percussion. Find V and ω at time of greatest pressure.

Problem 489. A man strikes a blow with a steel rod $1\frac{1}{4}$ in. in diameter and 4 ft. long, by holding the rod in the hand and striking the farther end against a stone in such a way as to cause the rod to be under flexure. Where should he grasp the rod in order that he may receive no shock?

Problem 490. Prove that after the blow has ceased to act on the body (Fig. 285) the body would then move in such a way that a circle of the body with radius CA and center C would roll along a straight line. Hence show that every point of the body in this circle would generate a cycloid. It is assumed that no other forces act on the body.

214. Oblique Impact of Body against Smooth Plane. — Let M (Fig. 286) be a sphere moving toward the plane indicated with a velocity at impact of v, the direction of motion making an angle α with the vertical to the plane. After im-

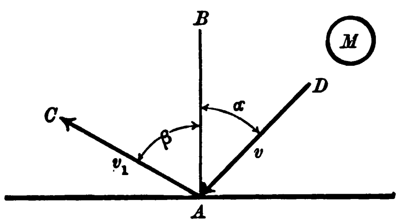


FIG. 286

pact the body M rebounds with a velocity v_1 in the direction making an angle β with the vertical AB. Since the plane is considered smooth, the effect of the impact will be all in the direction of AB, and hence the component of the velocity parallel to the plane will not change.

$$\therefore\ v_1 \sin \beta = v \sin \alpha. \tag{1}$$

Assuming that the plane does not move, $V = 0$, where V is the component of the velocity of M perpendicular to the plane at the time of greatest pressure. Then

$$\int_0^T P dt = M[0 - (v \cos \alpha)],$$

and

$$\int_T^{T'} R dt = M(-v_1 \cos \beta - 0).$$

Therefore

$$\frac{v_1 \cos \beta}{v \cos \alpha} = \frac{\int_T^{T'} R dt}{\int_0^T P dt} = e,$$

or

$$v_1 \cos \beta = ev \cos \alpha. \tag{2}$$

Dividing (1) by (2),

$$\tan \beta = \frac{1}{e} \tan \alpha.$$

Squaring and adding (1) and (2),

$$v_1^2 = v^2 (\sin^2 \alpha + e^2 \cos^2 \alpha).$$

Hence if α, v, and e are known, v_1 and β may be determined.

If the bodies are perfectly elastic, $e = 1$, $\beta = \alpha$, and $v_1 = v$. If the body is inelastic, $e = 0$, $\beta = \frac{\pi}{2}$, $v_1 = v \sin \alpha$.

The body then moves along the plane with a velocity $v \sin \alpha$.

Problem 491. A ball is projected with velocity 50 f/s against a smooth plane surface at an angle with the normal of 40°. It leaves at an angle of 50° with the normal. Find e and the velocity at which it leaves.

215. Impact of Rotating Bodies. — Suppose two bodies M_1 and M_2 revolve about two parallel axes O_1 and O_2 (Fig. 287) in such a way that impact occurs at a point along the line DE. Let the line along which impact occurs be distant r_1 and r_2 respectively from O_1 and O_2. Let P be the force of impact at any instant during the first period

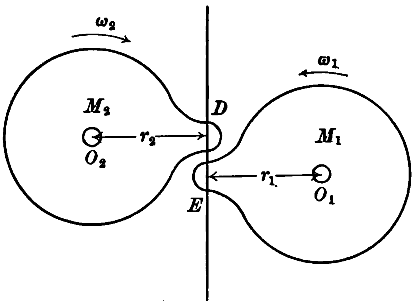


FIG. 287

and R the force at any instant during the second period, ω_1 and ω_2 the angular velocities at the beginning, ω_1' and ω_2' the angular velocities at the end of impact, I_1 and I_2 the moments of inertia of the bodies respectively about O_1 and O_2, and V the rectangular component of the velocity of the point of impact in the direction of DE at the time of greatest pressure. The angular velocities of the bodies at this instant are then $\frac{V}{r_1}$ and $\frac{V}{r_2}$.

Then, as in Art. 212,

$$r_1\int_0^T Pdt = I_1\int_{\omega_1}^{\frac{V}{r_1}} d\omega = I_1\left(\frac{V}{r_1} - \omega_1\right), \qquad (1)$$

$$-r_2\int_0^T Pdt = I_2\int_{\omega_2}^{\frac{V}{r_2}} d\omega = I_2\left(\frac{V}{r_2} - \omega_2\right), \tag{2}$$

$$r_1\int_T^{T'} Rdt = I_1\int_{\frac{V}{r_1}}^{\omega_1'} d\omega = I_1\left(\omega_1' - \frac{V}{r_1}\right), \tag{3}$$

$$-r_2\int_T^{T'} Rdt = I_2\int_{\frac{V}{r_2}}^{\omega_2'} d\omega = I_2\left(\omega_2' - \frac{V}{r_2}\right), \tag{4}$$

$$\int_T^{T'} Rdt = e\int_0^T Pdt. \tag{5}$$

From (1) and (2),

$$\frac{I_1}{r_1^2}(V - r_1\omega_1) + \frac{I_2}{r_2^2}(V - r_2\omega_2) = 0,$$

or
$$V = \frac{r_1 r_2(I_1 r_2\omega_1 + I_2 r_1\omega_2)}{I_1 r_2^2 + I_2 r_1^2}.$$

From (1), (3), and (5),

$$\omega_1' - \frac{V}{r_1} = e\left(\frac{V}{r_1} - \omega_1\right),$$

or
$$\omega_1' = \frac{V}{r_1}(1 + e) - e\omega_1.$$

Similarly,
$$\omega_2' = \frac{V}{r_2}(1 + e) - e\omega_2.$$

(It should be noted that in these formulæ the angular velocities of the bodies are reckoned in opposite directions.)

Problem 492. If $I_1 = 3000$, $I_2 = 15{,}000$, $\omega_1 = 1$ radian per second, $\omega_2 = 0$, $r_1 = 2$ ft., $r_2 = 3$ ft., and $e = 0$, find V, ω_1', ω_2', and the kinetic energy lost in impact.

Problem 493. A well drill is shown in principle in Fig. 288. The drill is supported by a cable that passes over a pulley C and is attached to a friction drum A. When A is held, the drill is raised by the operation of M_1 and M_2. Suppose that I is 300 and $\omega_1 = 3$

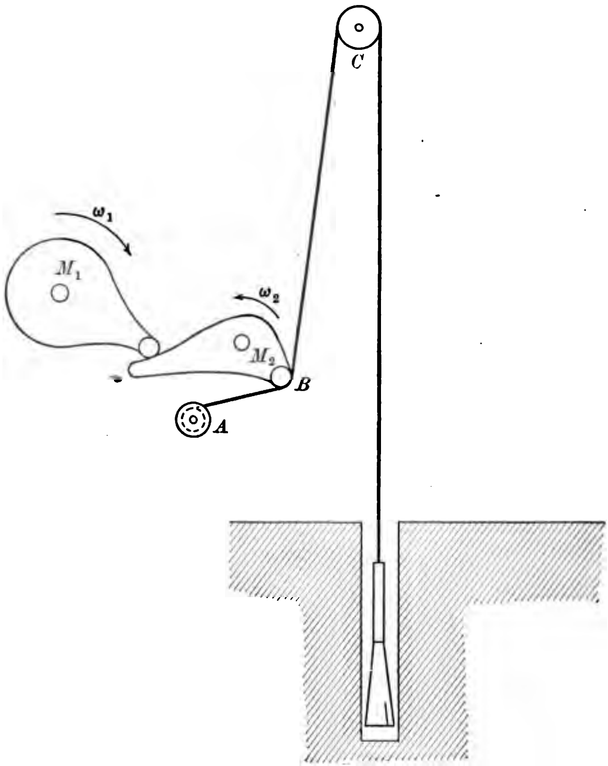


FIG. 288

radians per second; $I_2 = 200$ and $\omega_2 = 0$; $r_1 = 2$ ft. and $r_2 = 6$ ft. Assume $e = \frac{1}{2}$. Find ω_1' and ω_2'. What kinetic energy is lost due to each impact?

Problem 494. The moment of inertia of the trip hammer M_2, illustrated in principle in Fig. 289, is 100,000; that of M_1 is 60,000.

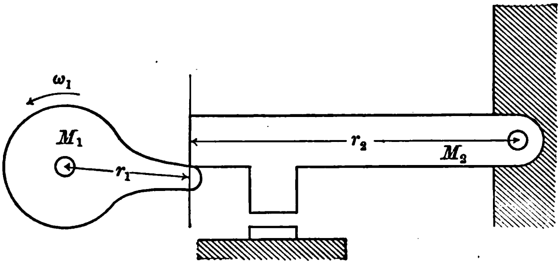


FIG. 289

If $r_1 = 3$ ft., $r_2 = 10$ ft., $\omega_1 = 2$ radians per second, $\omega_2 = 0$, and $e = \frac{1}{2}$, find ω_1' and ω_2'. What is the kinetic energy lost due to each impact? What is the kinetic energy of the hammer?

APPENDIX I

HYPERBOLIC FUNCTIONS

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

1

x	Cosh *x*	Sinh *x*	*x*	Cosh *x*	Sinh *x*
0.01	1.0000500	0.0100002	0.51	1.1328934	0.5323978
.02	.0002000	.0200013	.52	.1382741	.5437536
.03	.0004500	.0300045	.53	.1437686	.5551637
.04	.0008000	.0400107	.54	.1493776	.5666292
.05	.0012503	.0500208	.55	.1551014	.5781516
.06	.0018006	.0600360	.56	.1609408	.5897317
.07	.0024510	.0700572	.57	.1668962	.6013708
.08	.0032017	.0800854	.58	.1729685	.6130701
.09	.0040527	.0901215	.59	.1791579	.6248305
.10	.0050042	.1001668	.60	.1854652	.6366536
.11	.0060561	.1102220	.61	.1918912	.6485402
.12	.0072086	.1202882	.62	.1984363	.6604917
.13	.0084618	.1303664	.63	.2051013	.6725093
.14	.0098161	.1404578	.64	.2118867	.6845942
.15	.0112711	.1505631	.65	.2187933	.6967475
.16	.0128274	.1606835	.66	.2258219	.7089704
.17	.0144849	.1708200	.67	.2329730	.7212643
.18	.0162438	.1809735	.68	.2402474	.7336303
.19	.0181044	.1911452	.69	.2476458	.7460697
.20	.0200668	.2013360	.70	.2551690	.7585837
.21	.0221311	.2115469	.71	.2628178	.7711735
.22	.0242977	.2217790	.72	.2705927	.7838405
.23	.0265668	.2320333	.73	.2784948	.7965858
.24	.0289384	.2423107	.74	.2865248	.8094107
.25	.0314132	.2526122	.75	.2946833	.8223167
.26	.0339908	.2629393	.76	.3029713	.8353049
.27	.0366720	.2732925	.77	.3113896	.8483766
.28	.0394568	.2836731	.78	.3199392	.8615330
.29	.0423456	.2940819	.79	.3286206	.8747758
.30	.0453385	.3045203	.80	.3374349	.8881060
.31	.0484361	.3149891	.81	.3463831	.9015249
.32	.0516384	.3254894	.82	.3554658	.9150342
.33	.0549460	.3360222	.83	.3646840	.9286347
.34	.0583590	.3465886	.84	.3740388	.9423282
.35	.0618778	.3571898	.85	.3835309	.9561160
.36	.0655029	.3678265	.86	.3931614	.9699993
.37	.0692345	.3785001	.87	.4029312	.9839796
.38	.0730730	.3892116	.88	.4128413	0.9980584
.39	.0770189	.3999619	.89	.4228927	1.0122369
.40	.0810724	.4107523	.90	.4330864	.0265167
.41	.0852341	.4215838	.91	.4434234	.0408991
.42	.0895042	.4324574	.92	.4539048	.0553856
.43	.0938888	.4433742	.93	.4645315	.0699777
.44	.0983718	.4543354	.94	.4753046	.0846768
.45	.1029702	.4653420	.95	.4862254	.0994843
.46	.1076788	.4763952	.96	.4972947	.1144018
.47	.1124983	.4874959	.97	.5085137	.1294307
.48	.1174289	.4986455	.98	.5198837	.1445726
.49	.1224712	.5098450	.99	.5314057	.1598288
0.50	1.1276260	0.5210953	1.00	1.5430806	1.1752012

2

x	Cosh x	Sinh x	x	Cosh x	Sinh x
1.01	1.5549100	1.1906910	1.51	2.3738201	2.1529104
1.02	.5668948	.2062999	1.52	.3954676	.1767566
1.03	.5790365	.2220294	1.53	.4173563	.2008206
1.04	.5913358	.2378812	1.54	.4394857	.2251046
1.05	.6037945	.2538567	1.55	.4618591	.2496111
1.06	.6164134	.2699576	1.56	.4844787	.2743426
1.07	.6291940	.2861855	1.57	.5073467	.2993014
1.08	.6421375	.3025420	1.58	.5304654	.3244903
1.09	.6552453	.3190288	1.59	.5538373	.3499117
1.10	.6685186	.3356474	1.60	.5774645	.3755679
1.11	.6819587	.3523997	1.61	.6013494	.4014618
1.12	.7005670	.3642872	1.62	.6254945	.4275958
1.13	.7093449	.3863116	1.63	.6499021	.4539726
1.14	.7232938	.4034746	1.64	.6745748	.4805947
1.15	.7374148	.4207781	1.65	.6995149	.5074650
1.16	.7517098	.4382235	1.66	.7247249	.5345859
1.17	.7661798	.4558128	1.67	.7502074	.5619603
1.18	.7808265	.4735477	1.68	.7759650	.5895910
1.19	.7956513	.4914299	1.69	.8020001	.6174806
1.20	.8106556	.5094613	1.70	.8283154	.6456319
1.21	.8258410	.5276436	1.71	.8549136	.6740479
1.22	.8412089	.5459788	1.72	.8817974	.7027311
1.23	.8567610	.5644685	1.73	.9089692	.7316847
1.24	.8724988	.5831146	1.74	.9364319	.7609115
1.25	.8884239	.6019191	1.75	.9641884	.7904143
1.26	.9045378	.6208837	1.76	2.9922411	.8201962
1.27	.9208421	.6400105	1.77	3.0205932	.8502601
1.28	.9373385	.6593012	1.78	.0492473	.8806091
1.29	.9540287	.6787578	1.79	.0782063	.9112461
1.30	.9709143	.6983824	1.80	.1074732	.9421742
1.31	1.9879969	.7181768	1.81	.1370508	2.9733966
1.32	2.0052783	.7381431	1.82	.1669421	3.0049163
1.33	.0227603	.7582830	1.83	.1971501	.0367365
1.34	.0404446	.7785989	1.84	.2276799	.0688603
1.35	.0583329	.7990926	1.85	.2585283	.1012911
1.36	.0764271	.8197662	1.86	.2897047	.1340321
1.37	.0947288	.8406219	1.87	.3212100	.1670863
1.38	.1132401	.8616615	1.88	.3530475	.2004573
1.39	.1319627	.8828874	1.89	.3852202	.2341484
1.40	.1508985	.9043015	1.90	.4177315	.2681629
1.41	.1700494	.9259060	1.91	.4505846	.3025041
1.42	.1894172	.9477032	1.92	.4837827	.3371758
1.43	.2090041	.9696951	1.93	.5173293	.3721810
1.44	.2288118	1.9918840	1.94	.5512275	.4075235
1.45	.2488424	2.0142721	1.95	.5854808	.4432067
1.46	.2690979	.0368616	1.96	.6200927	.4792343
1.47	.2895803	.0596549	1.97	.6550667	.5156097
1.48	.3102917	.0826540	1.98	.6904061	.5523368
1.49	.3312341	.1058614	1.99	.7261146	.5894191
1.50	2.3524096	2.1292794	2.00	3.7621957	3.6268604

3

x	Cosh x	Sinh x	x	Cosh x	Sinh x
2.01	3.7986528	3.6646642	2.51	6.1930993	6.1118311
2.02	.8354899	.7028345	2.52	.2545281	.1740685
2.03	.8727101	.7413746	2.53	.3165827	.2369237
2.04	.9103184	.7802896	2.54	.3792687	.3004023
2.05	.9483548	.8196198	2.55	.4425928	.3645111
2.06	.9867111	.8592571	2.56	.5065611	.4292563
2.07	4.0255038	.8993179	2.57	.5711800	.4946444
2.08	.0647395	.9398093	2.58	.6364560	.5606820
2.09	.1043012	.9806140	2.59	.7023958	.6273758
2.10	.1443131	4.0218567	2.60	.7690059	.6947323
2.11	.1847398	.0635018	2.61	.8362940	.7627595
2.12	.2255846	.1055530	2.62	.9042644	.8314615
2.13	.2668523	.1480149	2.63	.9729254	.9008469
2.14	.3085462	.1908914	2.64	7.0422838	.9709225
2.15	.3506713	.2341871	2.65	.1123463	7.0416950
2.16	.3932312	.2779062	2.66	.1831184	.1131701
2.17	.4362311	.3220534	2.67	.2546108	.1853586
2.18	.4796741	.3666325	2.68	.3268282	.2582650
2.19	.5235649	.4116482	2.69	.3997785	.3318975
2.20	.5679083	.4571052	2.70	.4734686	.4062631
2.21	.6127086	.5030079	2.71	.5479060	.4813692
2.22	.6579702	.5493610	2.72	.6230984	.5572237
2.23	.7036972	.5961688	2.73	.6990531	.6338338
2.24	.7498951	.6434364	2.74	.7757775	.7112072
2.25	.7965677	.6911685	2.75	.8532799	.7893520
2.26	.8437197	.7393692	2.76	.9315674	.8682756
2.27	.8913565	.7880444	2.77	8.0106482	.9479862
2.28	.9394824	.8371982	2.78	.0905297	8.0284911
2.29	.9881022	.8868358	2.79	.1712205	.1097993
2.30	5.0372206	.9369618	2.80	.2527285	.1919185
2.31	.0868429	.9875817	2.81	.3350617	.2748566
2.32	.1369741	5.0387004	2.82	.4182283	.3586224
2.33	.1876186	.0903228	2.83	.5022368	.4432239
2.34	.2387822	.1424545	2.84	.5870956	.5286699
2.35	.2905196	.1951504	2.85	.6728130	.6149687
2.36	.3426859	.2482656	2.86	.7593979	.7021291
2.37	.3954365	.3019558	2.87	.8468585	.7901595
2.38	.4487266	.3561760	2.88	.9352041	.8790694
2.39	.5025618	.4109321	2.89	9.0244430	.9688668
2.40	.5569472	.4662293	2.90	.1145844	9.0595611
2.41	.6118883	.5220729	2.91	.2056373	.1511616
2.42	.6673910	.5784683	2.92	.2976105	.2436769
2.43	.7234594	.6354226	2.93	.3905138	.3371168
2.44	.7801009	.6929401	2.94	.4843559	.4314902
2.45	.8373201	.7510265	2.95	.5791467	.5268070
2.46	.8951232	.8096882	2.96	.6748952	.6230763
2.47	.9535159	.8689310	2.97	.7716115	.7203081
2.48	6.0125038	.9287605	2.98	.8693047	.8185119
2.49	.0720930	.9891831	2.99	.9679850	.9176976
2.50	6.1322895	6.0502045	3.00	10.0676620	10.0178750

4

x	Cosh x	Sinh x	x	Cosh x	Sinh x
3.01	10.1683456	10.1190539	3.51	16.7390823	16.7091854
3.02	.2700464	.2212451	3.52	.9070139	.8774144
3.03	.3727741	.3244585	3.53	17.0766361	17.0473312
3.04	.4765391	.4287042	3.54	.2479662	.2189529
3.05	.5813518	.5339929	3.55	.4210213	.3922966
3.06	.6872224	.6403347	3.56	.5958178	.5673790
3.07	.7941620	.7477408	3.57	.7723744	.7442186
3.08	.9021809	.8562217	3.58	.9507082	.9228325
3.09	11.0112900	.9657881	3.59	18.1308371	18.1032388
3.10	.1215004	11.0764511	3.60	.3127790	.2854552
3.11	.2328226	.1882217	3.61	.4965523	.4695004
3.12	.3452684	.3011112	3.62	.6821753	.6553927
3.13	.4588488	.4151309	3.63	.8696665	.8431503
3.14	.5735748	.5302919	3.64	19.0590447	19.0327924
3.15	.6894584	.6466062	3.65	.2503288	.2243376
3.16	.8065107	.7640850	3.66	.4435377	.4178052
3.17	.9247440	.8827403	3.67	.6386909	.6132145
3.18	12.0441695	12.0025838	3.68	.8358083	.8105854
3.19	.1647998	.1236279	3.69	20.0349094	20.0099373
3.20	.2866462	.2458839	3.70	.2360140	.2112905
3.21	.4097213	.3693646	3.71	.4391421	.4146645
3.22	.5340375	.4940825	3.72	.6443142	.6200802
3.23	.6596073	.6200497	3.73	.8515505	.8275577
3.24	.7864428	.7472790	3.74	21.0608720	21.0371178
3.25	.9145572	.8757829	3.75	.2722997	.2487819
3.26	13.0439629	13.0055744	3.76	.4858548	.4625710
3.27	.1746730	.1366665	3.77	.7015584	.6785064
3.28	.3067006	.2690723	3.78	.9194324	.8966096
3.29	.4400587	.4028048	3.79	22.1394981	22.1169025
3.30	.5747611	.5378780	3.80	.3617777	.3394069
3.31	.7108208	.6743046	3.81	.5862933	.5641452
3.32	.8482516	.8120988	3.82	.8130681	.7911403
3.33	.9870673	.9512741	3.83	23.0421239	23.0204143
3.34	14.1272820	14.0918450	3.84	.2734843	.2519907
3.35	.2689091	.2338247	3.85	.5071715	.4858917
3.36	.4119630	.3772277	3.86	.7432095	.7221415
3.37	.5564583	.5220686	3.87	.9816222	.9607638
3.38	.7024094	.6683619	3.88	24.2224327	24.2017819
3.39	.8498306	.8161219	3.89	.4656658	.4452205
3.40	.9987366	.9653634	3.90	.7113454	.6911034
3.41	15.1491429	15.1161016	3.91	.9594963	.9394557
3.42	.3010637	.2683513	3.92	25.2101431	25.1903020
3.43	.4545147	.4221278	3.93	.4633109	.4436673
3.44	.6095114	.5774468	3.94	.7190247	.6995765
3.45	.7660688	.7343232	3.95	.9773109	.9580561
3.46	.9242033	.8927735	3.96	26.2381943	26.2191311
3.47	16.0839298	16.0528128	3.97	.5017019	.4828285
3.48	.2452646	.2144571	3.98	.7678597	.7491740
3.49	.4082241	.3777233	3.99	27.0366943	27.0181946
3.50	16.5728248	16.5426275	4.00	.3082331	.2899175

APPENDIX II

LOGARITHMS OF NUMBERS

1

LOGARITHMS OF NUMBERS, FROM 0 TO 1000

No.	0	1	2	3	4	5	6	7	8	9
0	0	00000	30103	47712	60206	69897	77815	84510	90309	95424
10	00000	00432	00860	01283	01703	02118	02530	02938	03342	03742
11	04139	04532	04921	05307	05690	06069	06445	06818	07188	07554
12	07918	08278	08636	08990	09342	09691	10037	10380	10721	11059
13	11394	11727	12057	12385	12710	13033	13353	13672	13987	14301
14	14613	14921	15228	15533	15836	16136	16435	16731	17026	17318
15	17609	17897	18184	18469	18752	19033	19312	19590	19865	20139
16	20412	20682	20951	21218	21484	21748	22010	22271	22530	22788
17	23045	23299	23552	23804	24054	24303	24551	24797	25042	25285
18	25527	25767	26007	26245	26481	26717	26951	27184	27415	27646
19	27875	28103	28330	28555	28780	29003	29225	29446	29666	29885
20	30103	30319	30535	30749	30963	31175	31386	31597	31806	32014
21	32222	32428	32633	32838	33041	33243	33445	33646	33845	34044
22	34242	34439	34635	34830	35024	35218	35410	35602	35793	35983
23	36173	36361	36548	36735	36921	37106	37291	37474	37657	37839
24	38021	38201	38381	38560	38739	38916	39093	39269	39445	39619
25	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330
26	41497	41664	41830	41995	42160	42324	42488	42651	42813	42975
27	43136	43296	43456	43616	43775	43933	44090	44248	44404	44560
28	44716	44870	45024	45178	45331	45484	45636	45788	45939	46089
29	46240	46389	46538	46686	46834	46982	47129	47275	47421	47567
30	47712	47856	48000	48144	48287	48430	48572	48713	48855	48995
31	49136	49276	49415	49554	49693	49831	49968	50105	50242	50379
32	50515	50650	50785	50920	51054	51188	51321	51454	51587	51719
33	51851	51982	52113	52244	52374	52504	52633	52763	52891	53020
34	53148	53275	53402	53529	53655	53781	53907	54033	54157	54282
35	54407	54530	54654	54777	54900	55022	55145	55266	55388	55509
36	55630	55750	55870	55990	56110	56229	56348	56466	56584	56702
37	56820	56937	57054	57170	57287	57403	57518	57634	57749	57863
38	57978	58092	58206	58319	58433	58546	58658	58771	58883	58995
39	59106	59217	59328	59439	59549	59659	59769	59879	59988	60097
40	60206	60314	60422	60530	60638	60745	60852	60959	61066	61172
41	61278	61384	61489	61595	61700	61804	61909	62013	62118	62221
42	62325	62428	62531	62634	62736	62838	62941	63042	63144	63245
43	63347	63447	63548	63648	63749	63848	63948	64048	64147	64246
44	64345	64443	64542	64640	64738	64836	64933	65030	65127	65224
45	65321	65417	65513	65609	65705	65801	65896	65991	66086	66181
46	66276	66370	66464	66558	66651	66745	66838	66931	67024	67117
47	67210	67302	67394	67486	67577	67669	67760	67851	67942	68033
48	68124	68214	68304	68394	68484	68574	68663	68752	68842	68930
49	69020	69108	69196	69284	69372	69460	69548	69635	69722	69810
50	69897	69983	70070	70156	70243	70329	70415	70500	70586	70671
51	70757	70842	70927	71011	71096	71180	71265	71349	71433	71516
52	71600	71683	71767	71850	71933	72015	72098	72181	72263	72345
53	72428	72509	72591	72672	72754	72835	72916	72997	73078	73158
54	73239	73319	73399	73480	73559	73639	73719	73798	73878	73957

2

LOGARITHMS OF NUMBERS, FROM 0 TO 1000
(Continued)

No.	0	1	2	3	4	5	6	7	8	9
55	74036	74115	74193	74272	74351	74429	74507	74585	74663	74741
56	74818	74896	74973	75050	75127	75204	75281	75358	75434	75511
57	75587	75663	75739	75815	75891	75966	76042	76117	76192	76267
58	76342	76417	76492	76566	76641	76715	76789	76863	76937	77011
59	77085	77158	77232	77305	77378	77451	77524	77597	77670	77742
60	77815	77887	77959	78031	78103	78175	78247	78318	78390	78461
61	78533	78604	78675	78746	78816	78887	78958	79028	79098	79169
62	79239	79309	79379	79448	79518	79588	79657	79726	79796	79865
63	79934	80002	80071	80140	80208	80277	80345	80413	80482	80550
64	80618	80685	80753	80821	80888	80956	81023	81090	81157	81224
65	81291	81358	81424	81491	81557	81624	81690	81756	81822	81888
66	81954	82020	82085	82151	82216	82282	82347	82412	82477	82542
67	82607	82672	82736	82801	82866	82930	82994	83058	83123	83187
68	83250	83314	83378	83442	83505	83569	83632	83695	83758	83821
69	83884	83947	84010	84073	84136	84198	84260	84323	84385	84447
70	84509	84571	84633	84695	84757	84818	84880	84941	85003	85064
71	85125	85187	85248	85309	85369	85430	85491	85551	85612	85672
72	85733	85793	85853	85913	85973	86033	86093	86153	86213	86272
73	86332	86391	86451	86510	86569	86628	86687	86746	86805	86864
74	86923	86981	87040	87098	87157	87215	87273	87332	87390	87448
75	87506	87564	87621	87679	87737	87794	87852	87909	87966	88024
76	88081	88138	88195	88252	88309	88366	88422	88479	88536	88592
77	88649	88705	88761	88818	88874	88930	88986	89042	89098	89153
78	89209	89265	89320	89376	89431	89487	89542	89597	89652	89707
79	89762	89817	89872	89927	89982	90036	90091	90145	90200	90254
80	90309	90363	90417	90471	90525	90579	90633	90687	90741	90794
81	90848	90902	90955	91009	91062	91115	91169	91222	91275	91328
82	91381	91434	91487	91540	91592	91645	91698	91750	91803	91855
83	91907	91960	92012	92064	92116	92168	92220	92272	92324	92376
84	92427	92479	92531	92582	92634	92685	92737	92788	92839	92890
85	92941	92993	93044	93095	93146	93196	93247	93298	93348	93399
86	93449	93500	93550	93601	93651	93701	93751	93802	93852	93902
87	93951	94001	94051	94101	94151	94200	94250	94300	94349	94398
88	94448	94497	94546	94596	94645	94694	94743	94792	94841	94890
89	94939	94987	95036	95085	95133	95182	95230	95279	95327	95376
90	95424	95472	95520	95568	95616	95664	95712	95760	95808	95856
91	95904	95951	95999	96047	96094	96142	96189	96236	96284	96331
92	96378	96426	96473	96520	96567	96614	96661	96708	96754	96801
93	96848	96895	96941	96988	97034	97081	97127	97174	97220	97266
94	97312	97359	97405	97451	97497	97543	97589	97635	97680	97726
95	97772	97818	97863	97909	97954	98000	98045	98091	98136	98181
96	98227	98272	98317	98362	98407	98452	98497	98542	98587	98632
97	98677	98721	98766	98811	98855	98900	98945	98989	99033	99078
98	99122	99166	99211	99255	99299	99343	99387	99431	99475	99519
99	99563	99607	99651	99694	99738	99782	99825	99869	99913	99956

APPENDIX III

TRIGONOMETRIC FUNCTIONS

1

NATURAL SINES, COSINES, TANGENTS, ETC.

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
0	0	.000000	Infinite	.000000	Infinite	1.00000	1.000000	0	**90**
	10	.002909	343.77516	.002909	343.77371	1.00000	.999996	50	
	20	.005818	171.88831	.005818	171.88540	1.00002	.999983	40	
	30	.008727	114.59301	.008727	114.58865	1.00004	.999962	30	
	40	.011635	85.945609	.011636	85.939791	1.00007	.999932	20	
	50	.014544	68.757360	.014545	68.750087	1.00011	.999894	10	
1	0	.017452	57.298688	.017455	57.289962	1.00015	.999848	0	**89**
	10	.020361	49.114062	.020365	49.103881	1.00021	.999793	50	
	20	.023269	42.975713	.023275	42.964077	1.00027	.999729	40	
	30	.026177	38.201550	.026186	38.188459	1.00034	.999657	30	
	40	.029085	34.382316	.029097	34.367771	1.00042	.999577	20	
	50	.031992	31.257577	.032009	31.241577	1.00051	.999488	10	
2	0	.034899	28.653708	.034921	28.636253	1.00061	.999391	0	**88**
	10	.037806	26.450510	.037834	26.431600	1.00072	.999285	50	
	20	.040713	24.562123	.040747	24.541758	1.00083	.999171	40	
	30	.043619	22.925586	.043661	22.903766	1.00095	.999048	30	
	40	.046525	21.493676	.046576	21.470401	1.00108	.998917	20	
	50	.049431	20.230284	.049491	20.205553	1.00122	.998778	10	
3	0	.052336	19.107323	.052408	19.081137	1.00137	.998630	0	**87**
	10	.055241	18.102619	.055325	18.074977	1.00153	.998473	50	
	20	.058145	17.198434	.058243	17.169337	1.00169	.998308	40	
	30	.061049	16.380408	.061163	16.349855	1.00187	.998135	30	
	40	.063952	15.636793	.064083	15.604784	1.00205	.997357	20	
	50	.066854	14.957882	.067004	14.924417	1.00224	.997763	10	
4	0	.069756	14.335587	.069927	14.300666	1.00244	.997564	0	**86**
	10	.072658	13.763115	.072851	13.726738	1.00265	.997357	50	
	20	.075559	13.234717	.075776	13.196888	1.00287	.997141	40	
	30	.078459	12.745495	.078702	12.706205	1.00309	.996917	30	
	40	.081359	12.291252	.081629	12.250505	1.00333	.996685	20	
	50	.084258	11.868370	.084558	11.826167	1.00357	.996444	10	
5	0	.087156	11.473713	.087489	11.430052	1.00382	.996195	0	**85**
	10	.090053	11.104549	.090421	11.059431	1.00408	.995937	50	
	20	.092950	10.758488	.093354	10.711913	1.00435	.995671	40	
	30	.095846	10.433431	.096289	10.385397	1.00463	.995396	30	
	40	.098741	10.127522	.099226	10.078031	1.00491	.995113	20	
	50	.101635	9.8391227	.102164	9.7881732	1.00521	.994822	10	
6	0	.104528	9.5667722	.105104	9.5143645	1.00551	.994522	0	**84**
	10	.107421	9.3091699	.108046	9.2553035	1.00582	.994214	50	
	20	.110313	9.0651512	.110990	9.0098261	1.00614	.993897	40	**83**
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 83° 40′ to 90° read from bottom of table upward.

2

NATURAL SINES, COSINES, TANGENTS, ETC.
(*Continued*)

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
6	30	.113203	8.8336715	.113936	8.7768874	1.00647	.993572	30	
	40	.116093	8.6137901	.116883	8.5555468	1.00681	.993238	20	
	50	.118982	8.4045586	.119833	8.3449558	1.00715	.992896	10	
7	0	.121869	8.2055090	.122785	8.1443464	1.00751	.992546	0	83
	10	.124756	8.0156450	.125738	7.9530224	1.00787	.992187	50	
	20	.127642	7.8344335	.128694	7.7703506	1.00825	.991820	40	
	30	.130526	7.6612976	.131653	7.5957541	1.00863	.991445	30	
	40	.133410	7.4957100	.134613	7.4287064	1.00902	.991061	20	
	50	.136292	7.3371909	.137576	7.2687255	1.00942	.990669	10	
8	0	.139173	7.1852965	.140541	7.1153697	1.00983	.990268	0	82
	10	.142053	7.0396220	.143508	6.9682335	1.01024	.989859	50	
	20	.144932	6.8997942	.146478	6.8269437	1.01067	.989442	40	
	30	.147809	6.7654691	.149451	6.6911562	1.01111	.989016	30	
	40	.150686	6.6363293	.152426	6.5605538	1.01155	.988582	20	
	50	.153561	6.5120812	.155404	6.4348428	1.01200	.988139	10	
9	0	.156434	6.3924532	.158384	6.3137515	1.01247	.987688	0	81
	10	.159307	6.2771933	.161368	6.1970279	1.01294	.987229	50	
	20	.162178	6.1660674	.164354	6.0844381	1.01342	.986762	40	
	30	.165048	6.0588980	.167343	5.9757644	1.01391	.986286	30	
	40	.167916	5.9553625	.170334	5.8708042	1.01440	.985801	20	
	50	.170783	5.8553921	.173329	5.7693688	1.01491	.985309	10	
10	0	.173648	5.7587705	.176327	5.6712818	1.01543	.984808	0	80
	10	.176512	5.6653331	.179328	5.5763786	1.01595	.984298	50	
	20	.179375	5.5749258	.182332	5.4845052	1.01649	.983781	40	
	30	.182236	5.4874043	.185339	5.3955172	1.01703	.983255	30	
	40	.185095	5.4026333	.188359	5.3092793	1.01758	.982721	20	
	50	.187953	5.3204860	.191363	5.2256647	1.01815	.982178	10	
11	0	.190809	5.2408431	.194380	5.1445540	1.01872	.981627	0	79
	10	.193664	5.1635924	.197401	5.0658352	1.01930	.981068	50	
	20	.196517	5.0886284	.200425	4.9894027	1.01989	.980500	40	
	30	.199368	5.0158317	.203452	4.9151570	1.02049	.979925	30	
	40	.202218	4.9451687	.206483	4.8430045	1.02110	.979341	20	
	50	.205065	4.8764907	.209518	4.7728568	1.02171	.978748	10	
12	0	.207912	4.8097343	.212557	4.7046301	1.02234	.978148	0	78
	10	.210756	4.7448206	.215599	4.6382457	1.02298	.977539	50	
	20	.213599	4.6816748	.218645	4.5736287	1.02362	.976921	40	
	30	.216440	4.6202263	.221695	4.5107085	1.02428	.976296	30	
	40	.219279	4.5604080	.224748	4.4494181	1.02494	.975662	20	
	50	.222116	4.5021565	.227806	4.3896940	1.02562	.975020	10	77
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 77° 10′ to 83° 30′ read from bottom of table upward.

3

NATURAL SINES, COSINES, TANGENTS, ETC.

(Continued)

°	'	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	'	°
13	0	.224951	4.4454115	.230868	4.3314759	1.02630	.974370	0	**77**
	10	.227784	4.3901158	.233934	4.2747066	1.02700	.973712	50	
	20	.230616	4.3362150	.237004	4.2193318	1.02770	.973045	40	
	30	.233445	4.2836576	.240079	4.1652998	1.02842	.972370	30	
	40	.236273	4.2323943	.243158	4.1125614	1.02914	.971687	20	
	50	.239098	4.1823785	.246241	4.0610700	1.02987	.970995	10	
14	0	.241922	4.1335655	.249328	4.0107809	1.03061	.970296	0	**76**
	10	.244743	4.0859130	.252420	3.9616518	1.03137	.969588	50	
	20	.247563	4.0393804	.255517	3.9136420	1.03213	.968872	40	
	30	.250380	3.9939292	.258618	3.8667131	1.03290	.968148	30	
	40	.253195	3.9495224	.261723	3.8208281	1.03363	.967415	20	
	50	.256008	3.9061250	.264834	3.7759519	1.03447	.966675	10	
15	0	.258819	3.8637033	.267949	3.7320508	1.03528	.965926	0	**75**
	10	.261628	3.8222251	.271069	3.6890927	1.03609	.965169	50	
	20	.264434	3.7816596	.274195	3.6470467	1.03691	.964404	40	
	30	.267238	3.7419775	.277325	3.6058835	1.03774	.963630	30	
	40	.270040	3.7031506	.280460	3.5655749	1.03858	.962849	20	
	50	.272840	3.6651518	.283600	3.5260938	1.03944	.962059	10	
16	0	.275637	3.6279553	.286745	3.4874144	1.04030	.961262	0	**74**
	10	.278432	3.5915363	.289896	3.4495120	1.04117	.960456	50	
	20	.281225	3.5558710	.293052	3.4123626	1.04206	.959642	40	
	30	.284015	3.5209365	.296214	3.3759434	1.04295	.958820	30	
	40	.286803	3.4867110	.299380	3.3402326	1.04385	.957990	20	
	50	.289589	3.4531735	.302553	3.3052091	1.04477	.957151	10	
17	0	.292372	3.4203036	.305731	3.2708526	1.04569	.956305	0	**73**
	10	.295152	3.3880820	.308914	3.2371438	1.04663	.955450	50	
	20	.297930	3.3564900	.312104	3.2040638	1.04757	.954588	40	
	30	.300706	3.3255095	.315299	3.1715948	1.04853	.953717	30	
	40	.303479	3.2951234	.318500	3.1397194	1.04950	.952838	20	
	50	.306249	3.2653149	.321707	3.1084210	1.05047	.951951	10	
18	0	.309017	3.2360680	.324920	3.0776835	1.05146	.951057	0	**72**
	10	.311782	3.2073673	.328139	3.0474915	1.05246	.950154	50	
	20	.314545	3.1791978	.331364	3.0178301	1.05347	.949243	40	
	30	.317305	3.1515453	.334595	2.9886850	1.05449	.948324	30	
	40	.320062	3.1243959	.337833	2.9600422	1.05552	.947397	20	
	50	.322816	3.0977363	.341077	2.9318885	1.05657	.946462	10	
19	0	.325568	3.0715535	.344328	2.9042109	1.05762	.945519	0	**71**
	10	.328317	3.0458352	.347585	2.8769970	1.05869	.944568	50	
	20	.331063	3.0205693	.350848	2.8502349	1.05976	.943609	40	**70**
°	'	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	'	°

For functions from 70° 40′ to 77° 0′ read from bottom of table upward.

4

NATURAL SINES, COSINES, TANGENTS, ETC.
(Continued)

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
19	30	.333807	2.9957443	.354119	2.8239129	1.06085	.942641	30	
	40	.336547	2.9713490	.357396	2.7980198	1.06195	.941666	20	
	50	.339285	2.9473724	.360680	2.7725448	1.06306	.940684	10	
20	0	.342020	2.9238044	.363970	2.7474774	1.06418	.939693	0	70
	10	.344752	2.9006346	.367268	2.7228076	1.06531	.938694	50	
	20	.347481	2.8778532	.370573	2.6985254	1.06645	.937687	40	
	30	.350207	2.8554510	.373885	2.6746215	1.06761	.936672	30	
	40	.352931	2.8334185	.377204	2.6510867	1.06878	.935650	20	
	50	.355651	2.8117471	.380530	2.6279121	1.06995	.934619	10	
21	0	.358368	2.7904281	.383864	2.6050891	1.07115	.933580	0	69
	10	.361082	2.7694532	.387205	2.5826094	1.07235	.932534	50	
	20	.363793	2.7488144	.390554	2.5604649	1.07356	.931480	40	
	30	.366501	2.7285038	.393911	2.5386479	1.07479	.930418	30	
	40	.369206	2.7085139	.397275	2.5171507	1.07602	.929348	20	
	50	.371908	2.6888374	.400647	2.4959661	1.07727	.928270	10	
22	0	.374607	2.6694672	.404026	2.4750869	1.07853	.927184	0	68
	10	.377302	2.6503962	.407414	2.4545061	1.07981	.926090	50	
	20	.379994	2.6316180	.410810	2.4342172	1.08109	.924989	40	
	30	.382683	2.6131259	.414214	2.4142136	1.08239	.923880	30	
	40	.385369	2.5949137	.417626	2.3944889	1.08370	.922762	20	
	50	.388052	2.5769753	.421046	2.3750372	1.08503	.921638	10	
23	0	.390731	2.5593047	.424475	2.3558524	1.08636	.920505	0	67
	10	.393407	2.5418961	.427912	2.3369287	1.08771	.919364	50	
	20	.396080	2.5247440	.431358	2.3182606	1.08907	.918216	40	
	30	.398749	2.5078428	.434812	2.2998425	1.09044	.917060	30	
	40	.401415	2.4911874	.438276	2.2816693	1.09183	.915896	20	
	50	.404078	2.4747726	.441748	2.2637357	1.09323	.914725	10	
24	0	.406737	2.4585933	.445229	2.2460368	1.09464	.913545	0	66
	10	.409392	2.4426448	.448719	2.2285676	1.09606	.912358	50	
	20	.412045	2.4269222	.452218	2.2113234	1.09750	.911164	40	
	30	.414693	2.4114210	.455726	2.1942997	1.09895	.909961	30	
	40	.417338	2.3961367	.459244	2.1774920	1.10041	.908751	20	
	50	.419980	2.3810650	.462771	2.1608958	1.10189	.907533	10	
25	0	.422618	2.3662016	.466308	2.1445069	1.10338	.906308	0	65
	10	.425253	2.3515424	.469854	2.1283213	1.10488	.905075	50	
	20	.427884	2.3370833	.473410	2.1123348	1.10640	.903834	40	
	30	.430511	2.3228205	.476976	2.0965436	1.10793	.902585	30	
	40	.433135	2.3087501	.480551	2.0809438	1.10947	.901329	20	
	50	.435755	2.2948685	.484137	2.0655318	1.11103	.900065	10	64
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 64°-10′ to 70°-30′ read from bottom of table upward.

5

NATURAL SINES, COSINES, TANGENTS, ETC.

(*Continued*)

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
26	0	.438371	2.2811720	.487733	2.0503038	1.11260	.898794	0	**64**
	10	.440984	2.2676571	.491339	2.0352565	1.11419	.897515	50	
	20	.443593	2.2543204	.494955	2.0203862	1.11579	.896229	40	
	30	.446198	2.2411585	.498582	2.0056897	1.11740	.894934	30	
	40	.448799	2.2281681	.502219	1.9911637	1.11903	.893633	20	
	50	.451397	2.2153460	.505867	1.9768050	1.12067	.892323	10	
27	0	.453990	2.2026893	.509525	1.9626105	1.12233	.891007	0	**63**
	10	.456580	2.1901947	.513195	1.9485772	1.12400	.889682	50	
	20	.459166	2.1778595	.516876	1.9347020	1.12568	.888350	40	
	30	.461749	2.1656806	.520567	1.9209821	1.12738	.887011	30	
	40	.464327	2.1536553	.524270	1.9074147	1.12910	.885664	20	
	50	.466901	2.1417808	.527984	1.8939971	1.13083	.884309	10	
28	0	.469472	2.1300545	.531709	1.8807265	1.13257	.882948	0	**62**
	10	.472038	2.1184737	.535547	1.8676003	1.13433	.881578	50	
	20	.474600	2.1070359	.539195	1.8546159	1.13610	.880201	40	
	30	.477159	2.0957385	.542956	1.8417409	1.13789	.878817	30	
	40	.479713	2.0845792	.546728	1.8290628	1.13970	.877425	20	
	50	.482263	2.0735556	.550515	1.8164892	1.14152	.876026	10	
29	0	.484810	2.0626653	.554309	1.8040478	1.14335	.874620	0	**61**
	10	.487352	2.0519061	.558118	1.7917362	1.14521	.873206	50	
	20	.489890	2.0412757	.561939	1.7795524	1.14707	.871784	40	
	30	.492424	2.0307720	.565773	1.7674940	1.14896	.870356	30	
	40	.494953	2.0203929	.569619	1.7555590	1.15085	.868920	20	
	50	.497479	2.0101362	.573478	1.7437453	1.15277	.867476	10	
30	0	.500000	2.0000000	.577350	1.7320508	1.15470	.866025	0	**60**
	10	.502517	1.9899822	.581235	1.7204736	1.15665	.864567	50	
	20	.505030	1.9800810	.585134	1.7090116	1.15861	.863102	40	
	30	.507538	1.9702944	.589045	1.6976631	1.16059	.861629	30	
	40	.510043	1.9606206	.592970	1.6864261	1.16259	.860149	20	
	50	.512543	1.9510577	.596908	1.6752988	1.16460	.858662	10	
31	0	.515038	1.9416040	.600861	1.6642795	1.16663	.857167	0	**59**
	10	.517529	1.9322578	.604827	1.6533663	1.16868	.855665	50	
	20	.520016	1.9230173	.608807	1.6425576	1.17075	.854156	40	
	30	.522499	1.9138809	.612801	1.6318517	1.17283	.852640	30	
	40	.524977	1.9048469	.616809	1.6212469	1.17493	.851117	20	
	50	.527450	1.8959138	.620832	1.6107417	1.17704	.849586	10	
32	0	.529919	1.8870799	.624869	1.6003345	1.17918	.848048	0	**58**
	10	.532384	1.8783438	.628921	1.5900238	1.18133	.846503	50	
	20	.534844	1.8697040	.632988	1.5798079	1.18350	.844951	40	**57**
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 57°-40′ to 64°-0′ read from bottom of table upward.

6

NATURAL SINES, COSINES, TANGENTS, ETC.

(*Continued*)

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
32	30	.537300	1.8611590	.637079	1.5696856	1.18569	.843391	30	
	40	.539751	1.8527073	.641167	1.5596552	1.18790	.841825	20	
	50	.542197	1.8443476	.645280	1.5497155	1.19012	.840251	10	
33	0	.544639	1.8360785	.649408	1.5398650	1.19236	.838671	0	**57**
	10	.547076	1.8278985	.653531	1.5301025	1.19463	.837083	50	
	20	.549509	1.8198065	.657710	1.5204261	1.19691	.835488	40	
	30	.551937	1.8118010	.661886	1.5108352	1.19920	.833886	30	
	40	.554360	1.8038809	.666077	1.5013282	1.20152	.832277	20	
	50	.556779	1.7960449	.670285	1.4919039	1.20386	830661	10	
34	0	.559193	1.7882916	.674509	1.4825610	1.20622	.829038	0	**56**
	10	.561602	1.7806201	.678749	1.4732983	1.20859	.827407	50	
	20	.564007	1.7730290	.683007	1.4641147	1.21099	.825770	40	
	30	.566406	1.7655173	.687281	1.4550090	1.21341	.824126	30	
	40	.568801	1.7580837	.691573	1.4459801	1.21584	.822475	20	
	50	.571191	1.7507273	.695881	1.4370268	1.21830	.820817	10	
35	0	.573576	1.7434468	.700208	1.4281480	1.22077	.819152	0	**55**
	10	.575957	1.7362413	.704552	1.4193427	1.22327	.817480	50	
	20	.578332	1.7291096	.708913	1.4106098	1.22579	.815801	40	
	30	.580703	1.7220508	.713293	1.4019483	1.22833	.814116	30	
	40	.583069	1.7150639	.717691	1.3933571	1.23089	.812423	20	
	50	.585429	1.7081478	.722108	1.3848355	1.23347	.810723	10	
36	0	.587785	1.7013016	.726543	1.3763810	1.23607	.809017	0	**54**
	10	.590136	1.6945244	.730996	1.3679959	1.23869	.807304	50	
	20	.592482	1.6878151	.735469	1.3596764	1.24134	.805584	40	
	30	.594823	1.6811730	.739961	1.3514224	1.24400	.803857	30	
	40	.597159	1.6745970	.744472	1.3432331	1.24669	.802123	20	
	50	.599489	1.6680864	.749003	1.3351075	1.24940	.800383	10	
37	0	.601815	1.6616401	.753554	1.3270448	1.25214	.798636	0	**53**
	10	.604136	1.6552575	.758125	1.3190441	1.25489	.796882	50	
	20	.606451	1.6489376	.762716	1.3111046	1.25767	.795121	40	
	30	.608761	1.6426796	.767627	1.3032254	1.26047	.793353	30	
	40	.611067	1.6364828	.771959	1.2954057	1.26330	.791579	20	
	50	.613367	1.6303462	.776612	1.2876447	1.26615	.789798	10	
38	0	.615661	1.6242692	.781286	1.2799416	1.26902	.788011	0	**52**
	10	.617951	1.6182510	.785981	1.2722957	1.27191	.786217	50	
	20	.620235	1.6122908	.790698	1.2647062	1.27483	.784416	40	
	30	.622515	1.6063879	.795436	1.2571723	1.27778	.782608	30	
	40	.624789	1.6005416	.800196	1.2496933	1.28075	.780794	20	
	50	.627057	1.5947511	.804080	1.2422685	1.28374	.778973	10	**51**
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 51°-10′ to 57°-30′ read from bottom of table upward.

7

NATURAL SINES, COSINES, TANGENTS, ETC.

(Continued)

°	′	Sine	Cosecant	Tangent	Cotangent	Secant	Cosine	′	°
39	0	.629320	1.5890157	.809784	1.2348972	1.28676	.777146	0	51
	10	.631578	1.5833318	.814612	1.2275786	1.28980	.775312	50	
	20	.633831	1.5777077	.819463	1.2203121	1.29287	.773472	40	
	30	.636078	1.5721337	.824336	1.2130970	1.29597	.771625	30	
	40	.638320	1.5666121	.829234	1.2059327	1.29909	.769771	20	
	50	.640557	1.5611424	.834155	1.1988184	1.30223	.767911	10	
40	0	.642788	1.5557238	.839100	1.1917536	1.30541	.766044	0	50
	10	.645013	1.5503558	.844069	1.1847376	1.30861	.764171	50	
	20	.647233	1.5450378	.849062	1.1777698	1.31183	.762292	40	
	30	.649448	1.5397690	.854081	1.1708496	1.31509	.760406	30	
	40	.651657	1.5345491	.859124	1.1639763	1.31837	.758514	20	
	50	.653861	1.5293773	.864193	1.1571495	1.32168	.756615	10	
41	0	.656059	1.5242531	.869287	1.1503684	1.32501	.754710	0	49
	10	.658252	1.5191759	.874407	1.1436326	1.32838	.752798	50	
	20	.660439	1.5141452	.879553	1.1369414	1.33177	.750880	40	
	30	.662620	1.5091605	.884725	1.1302944	1.33519	.748956	30	
	40	.664796	1.5042211	.889924	1.1236909	1.33864	.747025	20	
	50	.666966	1.4993267	.895151	1.1171305	1.34212	.745088	10	
42	0	.669131	1.4944765	.900404	1.1106125	1.34563	.743145	0	48
	10	.671289	1.4896703	.905685	1.1041365	1.34917	.741195	50	
	20	.673443	1.4849073	.910994	1.0977020	1.35274	.739239	40	
	30	.675590	1.4801872	.916331	1.0913085	1.35634	.737277	30	
	40	.677732	1.4755095	.921697	1.0849554	1.35997	.735309	20	
	50	.679868	1.4708736	.927091	1.0786423	1.36363	.733335	10	
43	0	.681998	1.4662792	.932515	1.0723687	1.36733	.731354	0	47
	10	.684123	1.4617257	.937968	1.0661341	1.37105	.729367	50	
	20	.686242	1.4572127	.943451	1.0599381	1.37481	.727374	40	
	30	.688355	1.4527397	.948965	1.0537801	1.37860	.725374	30	
	40	.690462	1.4483063	.954508	1.0476598	1.38242	.723369	20	
	50	.692563	1.4439120	.960083	1.0415767	1.38628	.721357	10	
44	0	.694658	1.4395565	.965689	1.0355303	1.39016	.719340	0	46
	10	.696748	1.4352393	.971326	1.0295203	1.39409	.717316	50	
	20	.698832	1.4309602	.976996	1.0235461	1.39804	.715286	40	
	30	.700909	1.4267182	.982697	1.0176074	1.40203	.713251	30	
	40	.702981	1.4225134	.988432	1.0117088	1.40606	.711209	20	
	50	.705047	1.4183454	.994199	1.0058348	1.41012	.709161	10	
45	0	.707107	1.4142136	1.000000	1.0000000	1.41421	.707107	0	45
°	′	Cosine	Secant	Cotangent	Tangent	Cosecant	Sine	′	°

For functions from 45°-0′ to 51°-0′ read from bottom of table upward.

APPENDIX IV

CONVERSION TABLES

1

TABLES FOR CONVERTING UNITED STATES WEIGHTS AND MEASURES

METRIC TO CUSTOMARY

WEIGHTS

No.	Milligrams to Grains	Grams to Troy Ounces	Grams to Avoirdupois Ounces	Kilograms to Avoirdupois Pounds	Tonnes to Net Tons of 2000 Pounds	Tonnes to Gross Tons of 2240 Pounds
1	.01543	.03215	.03527	2.20462	1.10231	.98421
2	.03086	.06430	.07055	4.40924	2.20462	1.96841
3	.04630	.09645	.10582	6.61387	3.30693	2.95262
4	.06173	.12860	.14110	8.81849	4.40924	3.93682
5	.07716	.16075	.17637	11.02311	5.51156	4.92103
6	.09259	.19290	.21164	13.22773	6.61387	5.90524
7	.10803	.22506	.24692	15.43236	7.71618	6.88944
8	.12346	.25721	.28219	17.63698	8.81849	7.87365
9	.13889	.28936	.31747	19.84160	9.92080	8.85785

1 Kilogram = 15432.35639 Grains

LINEAR MEASURE

No.	Millimeters to 64ths of an Inch	Centimeters to Inches	Meters to Feet	Meters to Yards	Kilometers to Statute Miles	Kilometers to Nautical Miles
1	2.51968	.39370	3.280833	1.093611	.62137	.53959
2	5.03936	.78740	6.561667	2.187222	1.24274	1.07919
3	7.55904	1.18110	9.842500	3.280833	1.86411	1.61878
4	10.07872	1.57480	13.123333	4.374444	2.48548	2.15837
5	12.59840	1.96850	16.404167	5.468056	3.10685	2.69796
6	15.11808	2.36220	19.685000	6.561667	3.72822	3.23756
7	17.63776	2.75590	22.965833	7.655278	4.34959	3.77715
8	20.15744	3.14960	26.246667	8.748889	4.97096	4.31674
9	22.67712	3.54330	29.527500	9.842500	5.59233	4.85633

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TABLES FOR CONVERTING UNITED STATES WEIGHTS AND MEASURES

CUSTOMARY TO METRIC

WEIGHTS

No.	Grains to Milligrams	Troy Ounces to Grams	Avoirdupois Ounces to Grams	Avoirdupois Pounds to Kilograms	Net Tons of 2000 Pounds to Tonnes	Gross Tons of 2240 Pounds to Tonnes
1	64.79892	31.10348	28.34953	.45359	.90718	1.01605
2	129.59784	62.20696	56.69905	.90718	1.81437	2.03209
3	194.39675	93.31044	85.04858	1.36078	2.72155	3.04814
4	259.19567	124.41392	113.39811	1.81437	3.62874	4.06419
5	323.99459	155.51740	141.74763	2.26796	4.53592	5.08024
6	388.79351	186.62088	170.09716	2.72155	5.44311	6.09628
7	453.59243	217.72437	198.44669	3.17515	6.35029	7.11233
8	518.39135	248.82785	226.79621	3.62874	7.25748	8.12838
9	583.19026	279.93133	255.14574	4.08233	8.16466	9.14442

1 Avoirdupois Pound = 453.5924277 Grams

LINEAR MEASURE

No.	64ths of an Inch to Millimeters	Inches to Centimeters	Feet to Meters	Yards to Meters	Statute Miles to Kilometers	Nautical Miles to Kilometers
1	.39688	2.54001	.304801	.914402	1.60935	1.85325
2	.79375	5.08001	.609601	1.828804	3.21869	3.70650
3	1.19063	7.62002	.914402	2.743205	4.82804	5.55975
4	1.58750	10.16002	1.219202	3.657607	6.43739	7.41300
5	1.98438	12.70003	1.524003	4.572009	8.04674	9.26625
6	2.38125	15.24003	1.828804	5.486411	9.65608	11.11950
7	2.77813	17.78004	2.133604	6.400813	11.26543	12.97275
8	3.17501	20.32004	2.438405	7.315215	12.87478	14.82600
9	3.57188	22.86005	2.743205	8.229616	14.48412	16.67925

1 Nautical Mile = 1853.25 Meters
1 Gunter's Chain = 20.1168 Meters
1 Fathom = 1.829 Meters

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